The Finite Element Method Theory in Depth

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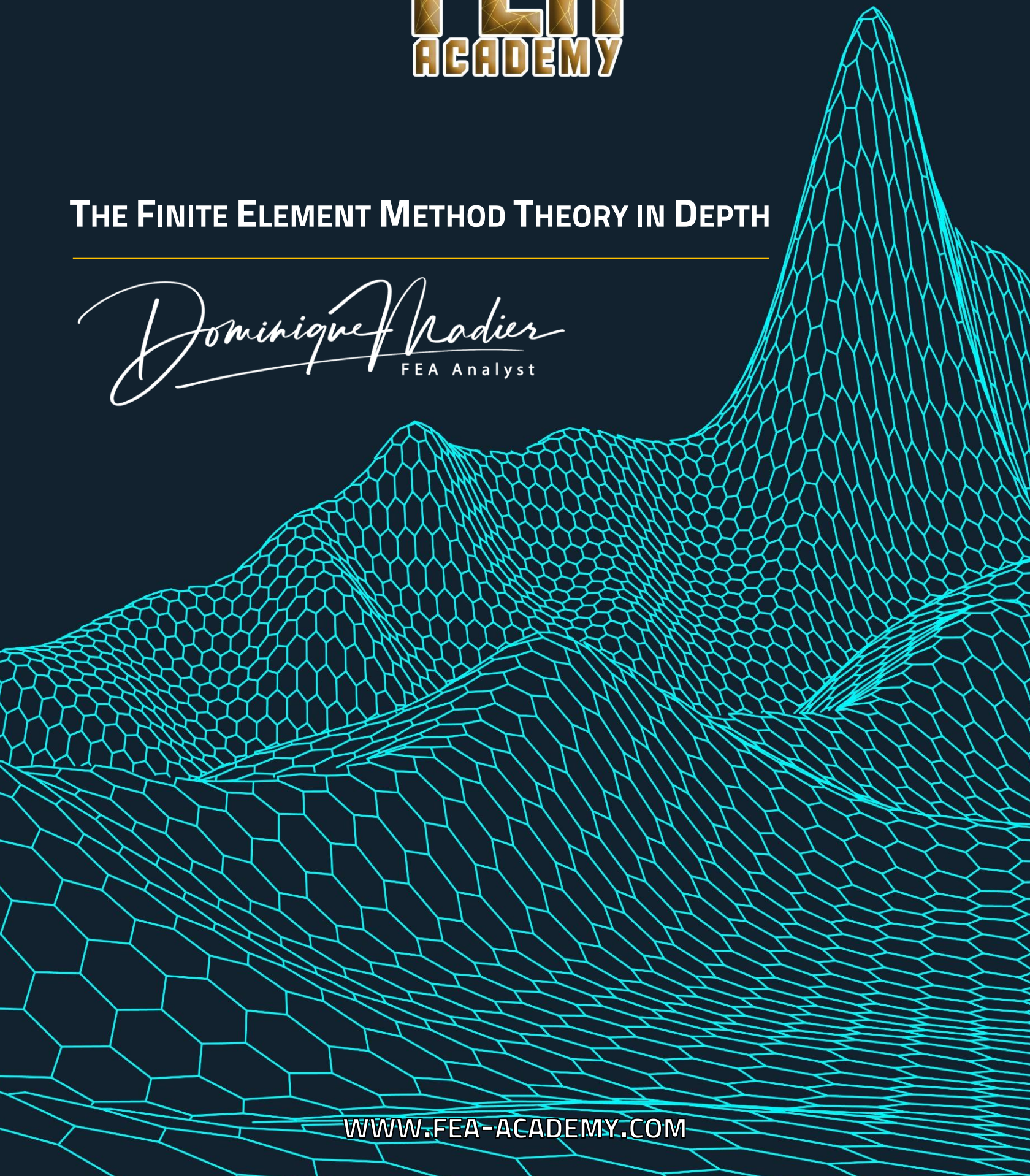

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THE FINITE ELEMENT METHOD THEORY IN DEPTH

Mathematics is not about numbers,
equations, computations, or algorithms:
It is about understanding.
William Paul Thurston

1. INTRODUCTION

As a mechanical engineer, you most likely will not develop your own finite element software, you will typically use commercial finite element software to solve complex structural problems. You may ask yourself why you should learn the theory of Finite Element Method (FEM).

It is certainly important to develop a practical understanding of FEA. However, it is also essential to understand exactly what the solver is doing in the background. If you have an issue with an analysis or an unexpected problem, you may need to understand what is happening in the background. This document therefore presents the theoretical aspect of the finite element method.

In chapter 5 of "Practical Finite Element Analysis for Mechanical Engineers", I present the Finite Element Method theory in fifty pages without showing all the demonstrations, just as an introduction. Here, I want to go deeper and not only present you with the results, but I also want to show you in detail where the things are coming from.

This document presents how to obtain the stiffness matrices of various topologies of elements. Then, I will present the matrix assembly and decomposition processes. Finally, I discuss the methods of solving simultaneous linear equations.

For a good understanding of stiffness matrix computation, you should know the basic equations of the theory of elasticity. They are presented in many texts so I will not discuss these equations. However, I recommend, especially, the book of Timoshenko and Goodier, *Theory of Elasticity, 3rd Edition, 1970.*

There are three basic sets of equations that the reader has to know:

1. The differential equations of equilibrium formulated in terms of the stresses acting on a body.
2. The strain-displacement and compatibility differential equations.
3. The stress-strain or material constitutive laws.

In Appendix C of "A First Course in the Finite Element Method, 4th Edition, 2006", Professor Daryl L. Logan did a remarkable and complete summary of these sets of equations. I will use these equations for the derivation of the element's stiffness matrices.

The theoretical concepts presented here are tackled step by step, from simplicity to complexity. If you are a beginner with the FEM theory, I strongly recommend you read this document from the beginning, following the concepts from the simplest topologies of 1D elements to the more complex 2D and 3D elements.

2. THE EQUILIBRIUM EQUATION

With the finite element method, a continuous system described by differential equations is idealized by a discrete system called the mesh. The discrete system is a collection of elements connected at nodes. A structural problem with an infinite number of degrees of freedom is converted into a problem with a finite number of degrees of freedom, making the problem solvable by a computer. When the nodes are displaced, the elements have an elastic response within their domain, providing a representation of the elastic properties of the complete system.

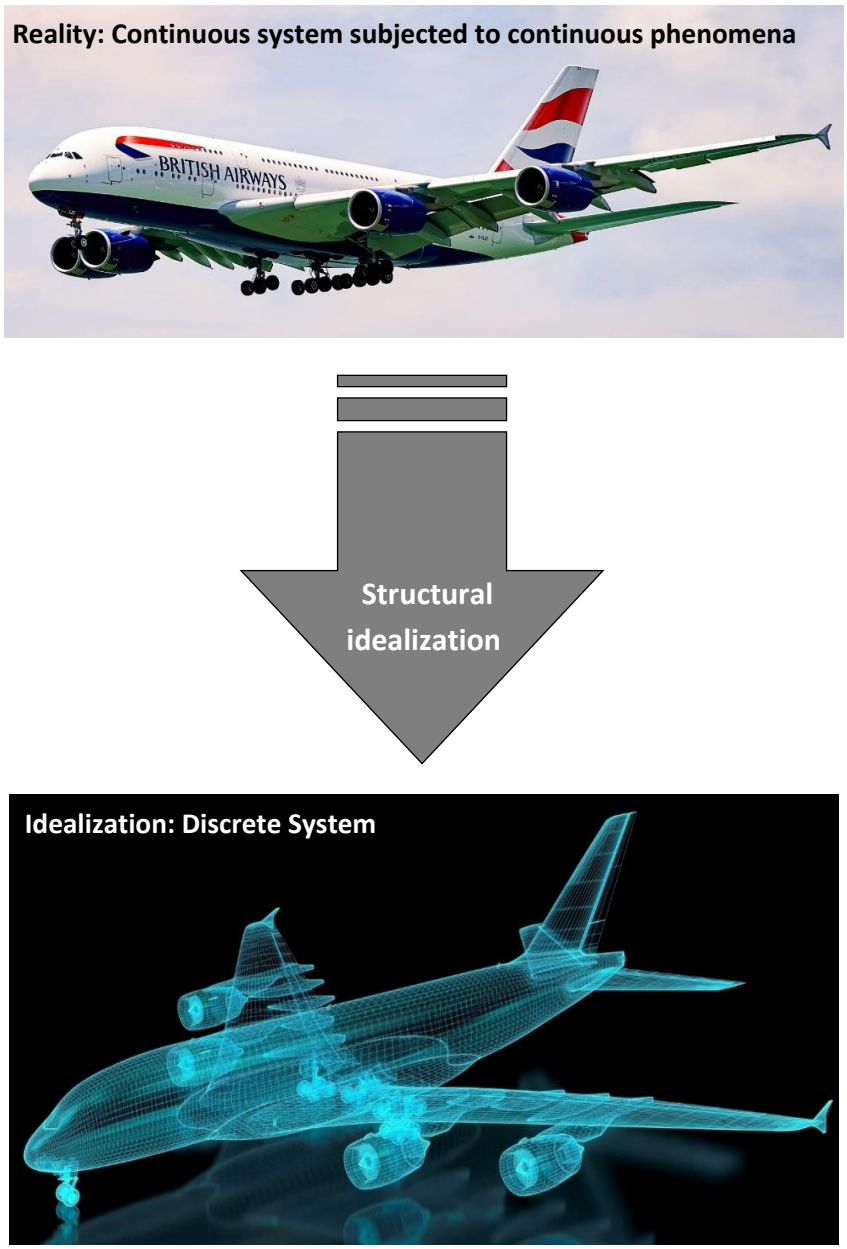


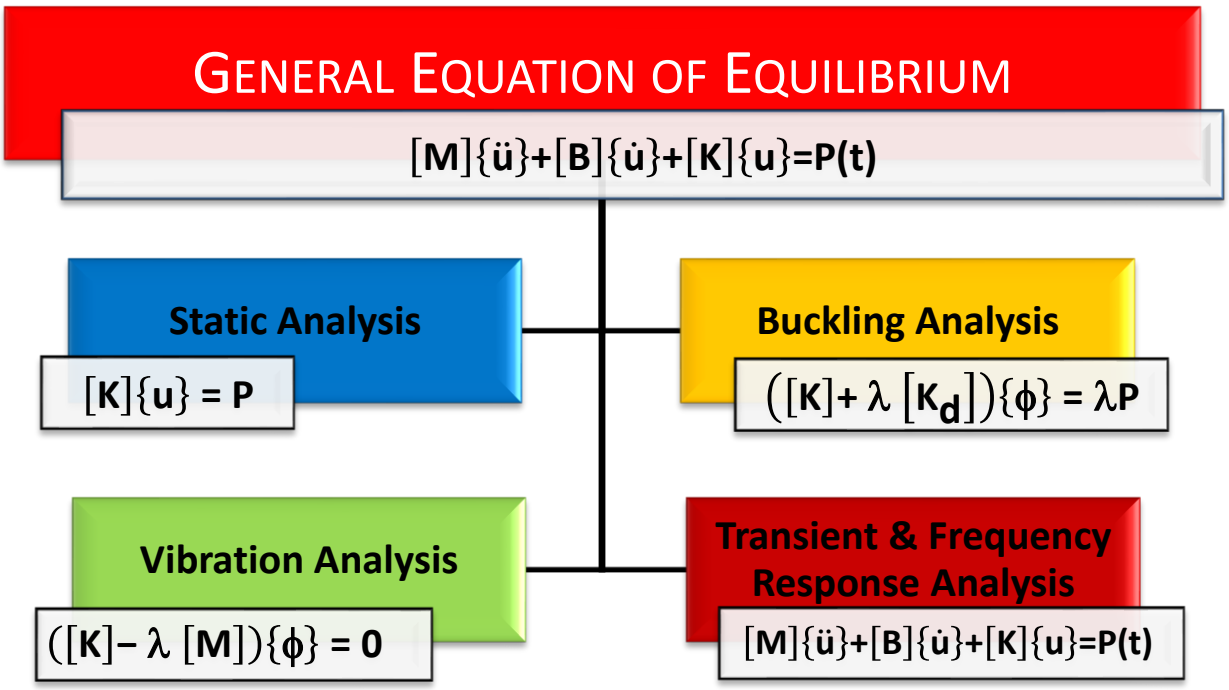
FIG 1 – FROM CONTINUOUS SYSTEM TO DISCRETE SYSTEM

For continuum mechanics problems, the unknown quantities of the mathematical model are the nodal displacements. The finite element method is based on the following matrix equation, which defines the state of equilibrium of forces acting on a structure:

$$[\boldsymbol{M}]\{\ddot{\boldsymbol{u}}\} + [\boldsymbol{B}]\{\dot{\boldsymbol{u}}\} + [\boldsymbol{K}]\{\boldsymbol{u}\} = \boldsymbol{P}(\boldsymbol{t}) \qquad \text{Eq 1}$$

[M]	Mass Matrix	{ü}	Acceleration Vector	P(t)	Applied Load
[B]	Damping Matrix	{u̇}	Velocity Vector		
[K]	Stiffness Matrix	{u}	Displacement Vector		

Variations of this equation of equilibrium are used to solve different types of structural problems:



λ is the ith eigenvalue.
$\{\Phi\}$ is the ith eigenvector.
$[K_d]$ is the Differential Stiffness Matrix (called also geometric stiffness).

FIG 2 - VARIATIONS OF THE GENERAL EQUATION OF EQUILIBRIUM

To solve structural problems, both the displacement method and the minimum potential energy approach are employed. We will have a look at these methods, and we will dive in detail into the formulation of the stiffness matrix for various topologies of elements. Finally, we will have a look at the process of matrix assembly and decomposition as well as the methods to solve the equation of equilibrium.

3. THE DISPLACEMENT METHOD

The displacement method (also called the stiffness method) assumes displacements at the nodes as the unknowns of the problem. I use the linear spring to explain the stiffness matrix concept, because it is a simple element that facilitates the illustration of basic concepts. I also present how to assemble the stiffness matrix of a system comprising several springs.

First, the following introduces three conditions that must always be satisfied in any analysis.

3.1. THE THREE CONDITIONS

In a linear static analysis, the following three conditions must be satisfied:

1. The equilibrium of forces and moments
2. The compatibility of deformations
3. The Stress-Strain relations

Equilibrium

Let recall the Newton's second law of motion:

- The resultant force acting on a body is equal to the rate of change of linear momentum (mass times velocity): $\sum F = \frac{d}{dt}(mv)$
- The resultant moment is equal to the rate of change of angular momentum (moment of inertia times angular velocity): $\sum M = \frac{d}{dt}(I\omega)$

A consequence of Newton's second law is that static equilibrium requires that the sum of forces and moments disappears at all points of the structure. Therefore, for a static body, the equilibrium equations are written as:

$$\sum \vec{F} = 0$$
$$\sum \vec{M} = 0 \qquad \text{Eq 2}$$

Compatibility of deformations

The compatibility of deformations (also called the strain-displacement relations) requires that the displacements are continuous across the body.

Stress-Strain relations

The generalized Hooke's law states that for a linear material:

$$\{\sigma\} = [E]\{\varepsilon\} \qquad \text{Eq 3}$$

Where:

- $\{\sigma\}=\{\sigma_x\ \sigma_y\ \sigma_z\ \tau_{xy}\ \tau_{yz}\ \tau_{zx}\}$ is the stress vector
- $\{\varepsilon\}=\{\epsilon_x\ \epsilon_y\ \epsilon_z\ \gamma_{xy}\ \gamma_{yz}\ \gamma_{zx}\}$ is the strain vector
- [E] is the 6x6 matrix of elastic constants.

3.2. THE STIFFNESS MATRIX

The three above conditions are used to generate a system of equations in which the displacements are unknown. In the displacement method, the key step is the formulation of the element stiffness matrix. The stiffness matrix [K] is used to relate the forces acting on the structure and the displacements resulting from these forces:

$$\{F\} = [K]\{u\} \qquad \text{Eq 4}$$

Where:

- {F} are the forces acting on the structure
- [K] is the stiffness matrix $[k_{ij}]$
- Each k_{ij} term is the force of a constraint at coordinate i due to a unit displacement at j with all other displacements set equal to zero
- {u} are the displacements resulting from {F}

Each element in a finite element model is represented by its own stiffness matrix $[K]_{element}$

To prevent rigid body motions, boundary conditions are applied, and the system of linear equations is solved for the unknown {u}.

3.2.1. Element Stiffness Matrix

The stiffness matrix of a structural system can be derived by various methods. For each element composing the structural mesh, the stiffness matrix is computed in the element local axis and is transformed to a global coordinate system common to all elements. The stiffness matrix is an inherent property of the structure. The properties of the stiffness matrix are:

- The stiffness matrix is square
- The stiffness matrix is symmetric
- All diagonal terms are positive

3.2.2. Global Stiffness Matrix

Once the element stiffness matrices are calculated, they are assembled into a unique global stiffness matrix depending on how the elements are connected. As we will see in paragraph 6 page 132, stiffness at a node is obtained by summing the stiffness of all elements connected to that node.

Assembling the global stiffness matrix of a structure therefore consists of placing each degree of freedom for each element in its proper position in the global stiffness matrix of the whole structure.

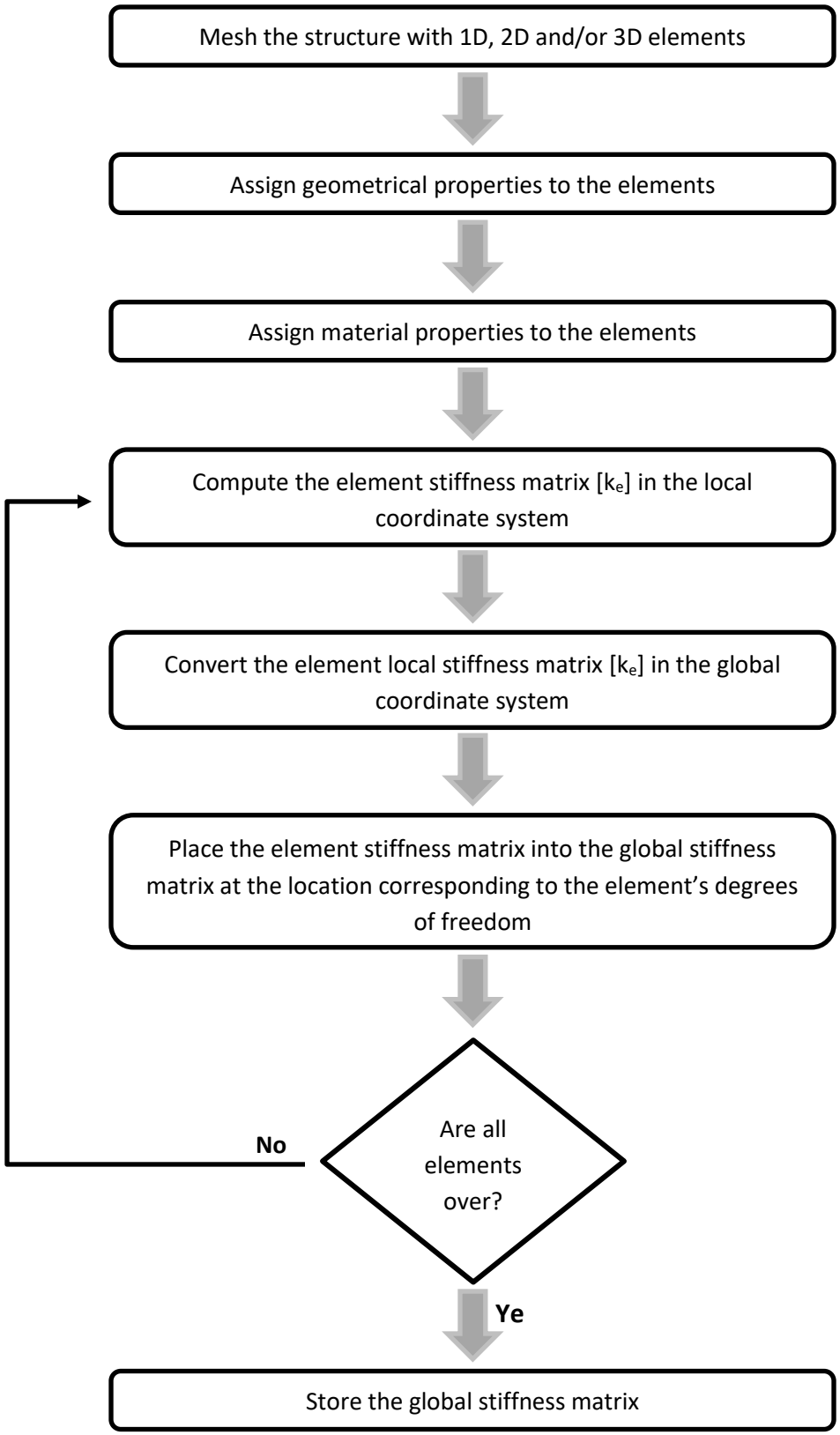


FIG 3 – ASSEMBLY OF STIFFNESS MATRIX FROM LOCAL TO GLOBAL

3.3. THE LINEAR SPRING MODEL

To understand the basic concepts on which the stiffness method is based, consider the simple, intuitive example of the linear spring. Consider the linear spring Fig 4 having a length L, a stiffness "k" and subjected to a nodal tensile force P along its axis.

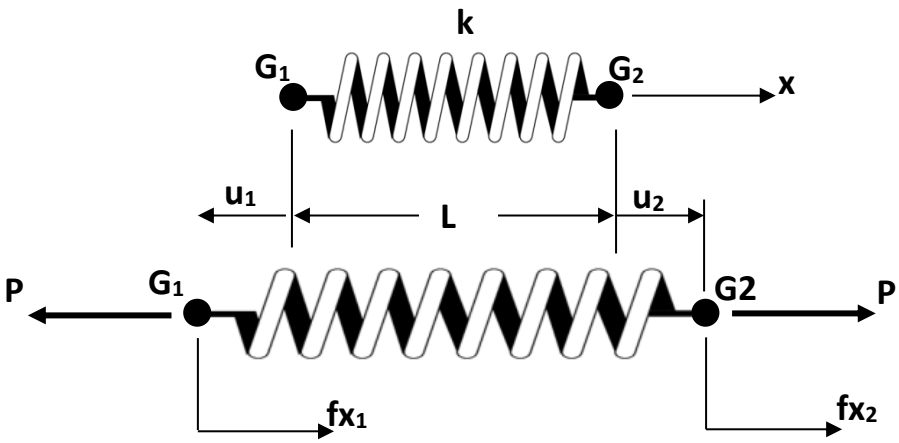


FIG 4 – THE LINEAR SPRING MODEL

The spring deformation is given as: $\delta = u_2 - u_1$

The Hooke's law relates the acting force P to the displacement: $P = K\delta = k(u_2 - u_1)$

Moreover, the equilibrium condition implies:

$fx_1 = -P$ and $fx_2 = P$

So, the force-displacements relationships are:

$fx_1 = -k(u_2 - u_1) = k(u_1 - u_2)$

$fx_2 = k(u_2 - u_1) = k(-u_1 + u_2)$

The force-displacements relationships can be written in matrix form:

$$\begin{Bmatrix} fx_1 \\ fx_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \text{Eq 5}$$

The coefficient matrix in the above equation is called the local stiffness matrix $[K]_{element}$ of the spring element:

$$[K]_{element} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \qquad \text{Eq 6}$$

The force vector acting on the element is:

$$[F]_{element} = \begin{Bmatrix} fx_1 \\ fx_2 \end{Bmatrix} \qquad \text{Eq 7}$$

Then, for a system discretized with N elements, the global stiffness matrix and the global force vector can be assembled using the nodal force equilibrium, force/deformation, and compatibility equations. The stiffness matrix of the system composed of N elements expressed in the global coordinates is then:

$$[K] = \sum_{e=1}^{N} [K]_e \qquad \text{Eq 8}$$

The force vector expressed in the global coordinates is:

$$[F] = \sum_{e=1}^{N} [F]_e \qquad \text{Eq 9}$$

Let's apply these results to a two-spring system in series.

3.4. APPLICATION TO THE TWO-SPRING SYSTEM

Consider the following two-spring system composed of two spring k1 and k2 subjected to a force P_2 at one extremity and a force P_3 in the middle. The extremity at grid 1 is fixed:

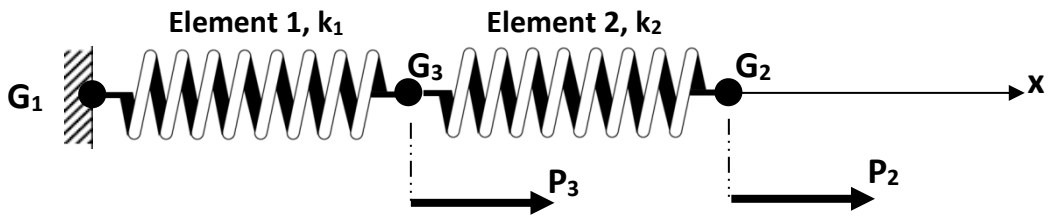


FIG 5 – THE TWO-SPRING SYSTEM

So, the following parameters,

- fx_1: force reacted at grid 1 of element 1
- fx_2: force reacted at grid 2 of element 2
- fx_3: force reacted at grid 3 common to elements 1 & 2
- u_1: displacement at grid 1 along the x-axis
- u_2: displacement at grid 2 along the x-axis
- u_3: displacement at grid 3 along the x-axis

can be related in matrix form for each element using the Eq 5:

For element 1:
$$\begin{Bmatrix} fx_1 \\ fx_3 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix} \qquad \text{Eq 10}$$

For element 2:
$$\begin{Bmatrix} fx_3 \\ fx_2 \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_2 \end{Bmatrix} \qquad \text{Eq 11}$$

The compatibility of displacements requires that: $u_3^{\text{element 1}} = u_3^{\text{element 2}}$

The nodal equilibrium equation at each node is then:

$$\begin{aligned} P_1 &= f_{X1}^{\text{element 1}} \\ P_2 &= f_{X2}^{\text{element 2}} \\ P_3 &= f_{X3}^{\text{element 1}} + f_{X3}^{\text{element 2}} \end{aligned} \qquad \text{Eq 12}$$

The three above equations can be written as force-displacement equations using Eq 10 & Eq 11:

$$\begin{aligned} P_1 &= k_1u_1 - k_1u_3 \\ P_2 &= -k_2u_3 + k_2u_2 \\ P_3 &= (-k_1u_1 + k_1u_3) + (k_2u_3 - k_2u_2) \end{aligned} \qquad \text{Eq 13}$$

The three above equations written in matrix form give:

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \text{Eq 14}$$

So, the global stiffness matrix of the two-spring system is:

$$[K] = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \qquad \text{Eq 15}$$

However, the global stiffness matrix can also be assembled by superposition.

Considering the two-spring system from Fig 5, the elemental stiffness matrix for each element can be written as:

<u>For element 1:</u>

$$[K]_{element\ 1} = \begin{array}{cc} \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} & \begin{array}{c} u_1 \\ u_3 \end{array} \\ \begin{array}{cc} u_1 & u_3 \end{array} & \end{array}$$

<u>For element 2:</u>

$$[K]_{element\ 2} = \begin{array}{cc} \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} & \begin{array}{c} u_3 \\ u_2 \end{array} \\ \begin{array}{cc} u_3 & u_2 \end{array} & \end{array}$$

If we write the *expanded stiffness matrix* in global format for each element, we obtain:

<u>For element 1:</u> $$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & 0 & 0 \\ -k_1 & 0 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_{x-Node\ 1}^{element\ 1} \\ 0 \\ f_{x-Node\ 3}^{element\ 1} \end{Bmatrix}$$

<u>For element 2:</u> $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_{x-Node\ 2}^{element\ 2} \\ f_{x-Node\ 3}^{element\ 2} \end{Bmatrix}$$

From the force equilibrium equations:

$$\begin{Bmatrix} f_{x-Node\ 1}^{element\ 1} \\ 0 \\ f_{x-Node\ 3}^{element\ 1} \end{Bmatrix} + \begin{Bmatrix} 0 \\ f_{x-Node\ 2}^{element\ 2} \\ f_{x-Node\ 3}^{element\ 2} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} \qquad \text{Eq 16}$$

We obtain,

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \text{Eq 17}$$

Which is the same result as Eq 14.

However, we can avoid the expansion of each element stiffness matrix by using a more direct form of the stiffness matrix. The global stiffness matrix may be constructed by directly adding terms associated with the degrees of freedom in $[K]_{element\ 1}$ and $[K]_{element\ 2}$ to their corresponding degrees of freedom:

For element 1:

$$[K]_{element\ 1} = \begin{array}{cc} \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} & \begin{array}{c} u_1 \\ u_3 \end{array} \\ \begin{array}{cc} u_1 & u_3 \end{array} & \end{array}$$

For element 2:

$$[K]_{element\ 2} = \begin{array}{cc} \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} & \begin{array}{c} u_3 \\ u_2 \end{array} \\ \begin{array}{cc} u_3 & u_2 \end{array} & \end{array}$$

$$[K] = \begin{array}{cc} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} & \\ \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} & \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \end{array} \qquad \text{Eq 18}$$

So, in order to solve the two-spring system:

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \text{Eq 19}$$

we have to verify if the stiffness matrix is not singular. In other words, we have to make sure that its determinant is not null.

$$det[K] = \begin{vmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{vmatrix} = k_1 \begin{vmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{vmatrix} - 0 \begin{vmatrix} 0 & -k_2 \\ -k_1 & k_1 + k_2 \end{vmatrix} - k_1 \begin{vmatrix} 0 & k_2 \\ -k_1 & -k_2 \end{vmatrix}$$

Eq 20

$$det[K] = k_1[k_2(k_1 + k_2) - k_2 k_2] - k_1(k_1 k_2) = k_1 k_2 k_1 - k_1 k_1 k_2 = 0 \qquad \text{Eq 21}$$

The global stiffness matrix of every system is singular because the structural system is free to move as a rigid body. To solve the system of equations, some constraints, called boundary conditions, must be applied. Two types can be applied: homogeneous and non-homogeneous boundary conditions.

Homogeneous boundary conditions

The boundary conditions are related to grid 1 which is clamped (homogeneous boundary condition). So, we can consider that $u_1 = 0$. The equations describing the elongation of the spring system become:

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \text{Eq 22}$$

For homogeneous boundary conditions, we can delete the row and column corresponding to the zero-displacement degree of freedom. The system becomes:

$$\begin{Bmatrix} P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \qquad \text{Eq 23}$$

The determinant of the above stiffness matrix is now non-null, so this system has a solution, u_2 and u_3 can be calculated. The reacted force P_1 at grid 1 can also be calculated from Eq 22:

$$P_1 = -k_1 u_3 \qquad \text{Eq 24}$$

Non-Homogeneous boundary conditions

Let's consider a non-homogeneous boundary condition at grid 1: $u_1 = \delta$

The equations describing the elongation of the spring system becomes:

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} \delta \\ u_2 \\ u_3 \end{Bmatrix} \qquad \text{Eq 25}$$

Where, δ is a known displacement and u_2 & u_3 are the unknown displacements.

For non-homogeneous boundary conditions, we cannot directly delete the row and column corresponding to the δ displacement degree of freedom, we have to transfer the terms from the stiffness matrix to the left-side force vector before solving the unknown displacements.

$$P_1 = k_1\delta - k_1 u_3$$

$$P_2 = k_2 u_2 - k_2 u_3 \qquad \text{Eq 26}$$

$$P_3 = -k_1\delta - k_2 u_2 + (k_1 + k_2) u_3$$

P_1 is a reaction from the support that has moved an amount of δ. The known δ term is moved on the left side:

$$\begin{Bmatrix} P_2 \\ P_3 + k_1\delta \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \qquad \text{Eq 27}$$

U_2 and U_3 can be calculated. Then the reacted force P_1 at grid 1 can also be calculated from:

$$P_1 = k_1\delta - k_1u_3.$$

3.5. APPLICATION TO THE FOUR-SPRING SYSTEM

Consider the following four-spring system:

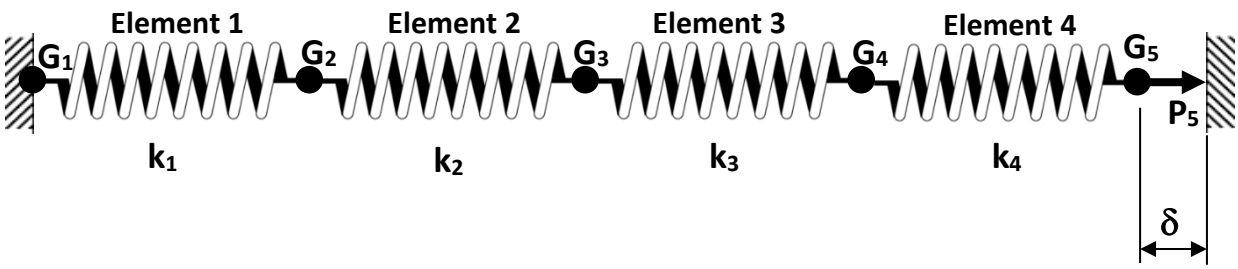


FIG 6 – THE FOUR-SPRING SYSTEM

The elemental stiffness matrices are:

$$[K]_{element\ 1} = \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \end{array} \qquad [K]_{element\ 2} = \begin{array}{c} \begin{array}{cc} u_2 & u_3 \end{array} \\ \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \end{array}$$

$$[K]_{element\ 3} = \begin{array}{c} \begin{array}{cc} u_3 & u_4 \end{array} \\ \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \end{array} \qquad [K]_{element\ 4} = \begin{array}{c} \begin{array}{cc} u_4 & u_5 \end{array} \\ \begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix} \end{array}$$

Using superposition, the global stiffness matrix is:

$$[K] = \begin{array}{c} \begin{array}{ccccc} \textit{Element 1} & \textit{Element 2} & \textit{Element 3} & \textit{Element 4} & \\ u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \end{array}$$

The global force-displacement equations are:

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} fx_1 \\ fx_2 \\ fx_3 \\ fx_4 \\ fx_5 \end{Bmatrix} \qquad \text{Eq 28}$$

Applying the boundary conditions (u_1=0 and u_5=δ) and the known forces (fx_2=fx_3=fx_4=0 & fx_5=P_5), we obtain:

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ u_4 \\ \delta \end{Bmatrix} = \begin{Bmatrix} fx_1 \\ 0 \\ 0 \\ 0 \\ P_5 \end{Bmatrix} \qquad \text{Eq 29}$$

$$\begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ \delta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_5 \end{Bmatrix} \qquad \text{Eq 30}$$

For non-homogeneous boundary conditions, we cannot directly delete the row and column corresponding to the δ displacement degree of freedom. We have to transfer the terms from the stiffness matrix to the right-side force vector before solving the unknown displacements. It gives the following solvable system:

$$\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ k_4\delta \end{Bmatrix} \qquad \text{Eq 31}$$

We can calculate the unknown displacements u_2, u_3 and u_4 from this system and also calculate the reacted force at grid 1 by $fx_1 = -k_1u_2$.

3.6. APPLICATION TO A PARALLEL-SPRING SYSTEM

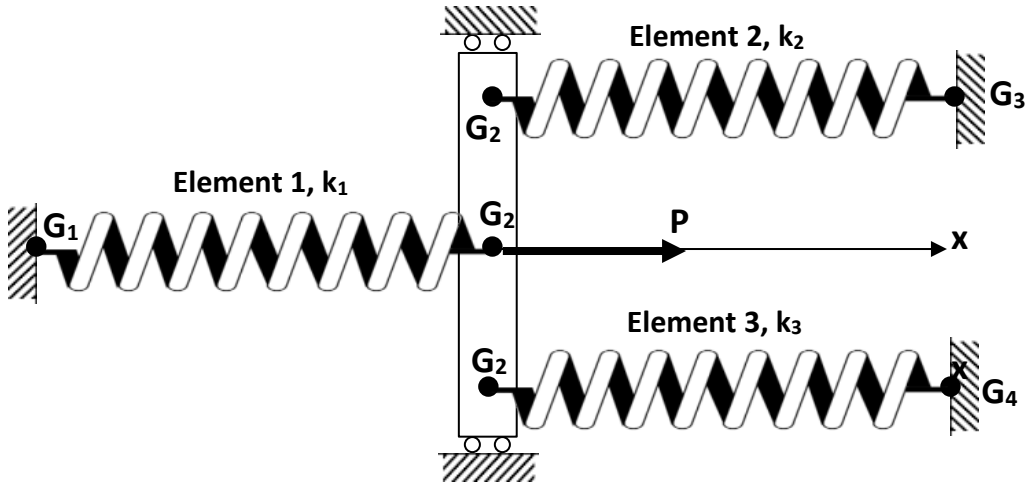


FIG 7 – THE PARALLEL-SPRING SYSTEM

The elemental stiffness matrices for each element are:

$$[K]_{element\ 1} = \begin{array}{cc} u_1 & u_2 \\ \end{array} \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{array}{c} u_1 \\ u_2 \end{array}$$

$$[K]_{element\ 2} = \begin{array}{cc} u_2 & u_3 \\ \end{array} \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{array}{c} u_2 \\ u_3 \end{array}$$

$$[K]_{element\ 3} = \begin{array}{cc} u_2 & u_4 \\ \end{array} \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \begin{array}{c} u_2 \\ u_4 \end{array}$$

Using the concept of superposition, the global stiffness matrix is:

$$[K] = \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \\ \end{array} \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \qquad \text{Eq 32}$$

Applying the boundary conditions: $u_1 = u_3 = u_4 = 0$

Applying the known force $fx_2 = P$

Applying the compatibility of displacements: $u_2^{element\ 1} = u_2^{element\ 2} = u_2^{element\ 3} = u_2$

The global force-displacement system becomes:

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} f_{x1} \\ P \\ f_{x3} \\ f_{x4} \end{Bmatrix} \qquad \text{Eq 33}$$

So, solving for the unknown displacement u_2:

$$(k_1 + k_2 + k_3)u_2 = P$$

$$u_2 = \frac{P}{k_1 + k_2 + k_3}$$

And, solving for the reacted forces gives:

$$fx_1 = -k_1u_2 \quad fx_3 = -k_2u_2 \quad fx_4 = -k_3u_2$$

4. THE PRINCIPLE OF MINIMUM POTENTIAL ENERGY

One of the alternative methods often used to derive the stiffness matrix for an element is based on the principle of minimum potential energy. This method has the advantage of being more general than the methods involving nodal and element equilibrium equations accommodated with the stress-strain law for the element. The principle of minimum potential energy is more adaptable for the determination of element equations for complicated elements with large numbers of degrees of freedom such as the plane stress-strain element, plate bending element, and solid element. Let's see how this principle works for the simple linear spring model.

The total potential energy π_p of a system is the sum of the internal strain energy U and the potential energy of the external forces W_e:

$$\pi_p = U + W_e \qquad \text{Eq 34}$$

- The Internal strain energy U represents the capacity of the internal forces to do work through deformation of the structure.
- The potential energy of the external forces W_e represents the capacity of the external forces to do work through deformation of the structure.

Reminder: the force-displacement relationship for a linear spring is F = kx

The differential internal work (or strain energy) dU in the spring is the internal force multiplied by the change in displacement:

$$dU = Fdx = (kx)dx \qquad \text{Eq 35}$$

The total strain energy is:

$$U = \int_0^X dU = \int_0^X (kx)dx = \frac{1}{2}kx^2 \qquad \text{Eq 36}$$

The potential energy of the external forces is the work done by the external forces, so: W = -Fx
Therefore, the total potential energy is:

$$\pi_p = \frac{1}{2}kx^2 - Fx \qquad \text{Eq 37}$$

Statically, to find the equilibrium of the spring corresponds to find a value of *x* yielding a stationary value of the potential energy π_p. This condition is obtained for:

$$\frac{\partial \pi_p}{\partial x} = 0 \qquad \text{Eq 38}$$

The variation of π_p can be written as:

$$\delta\pi_p = \frac{\partial\pi_p}{\partial u_1}\delta u_1 + \frac{\partial\pi_p}{\partial u_2}\delta u_2 + \cdots + \frac{\partial\pi_p}{\partial u_n}\delta u_n \qquad \text{Eq 39}$$

To satisfy $\partial\pi_p/\partial x = 0$, all coefficients associated with δu_i must be null independently:

$$\frac{\partial\pi_p}{\partial u_i} = 0 \quad for\ i = 1\ to\ n \qquad \text{Eq 40}$$

Consider the linear spring subjected to nodal forces from Fig 8 and let's derive the spring stiffness matrix using the principle of minimum potential energy:

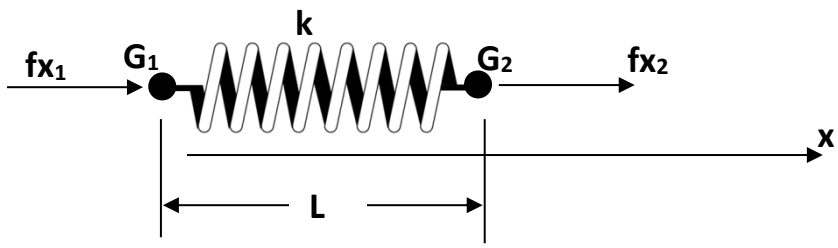


FIG 8 – THE LINEAR SPRING SYSTEM

The total potential energy π_p of the above spring is then:

$$\begin{aligned}\pi_p &= \frac{1}{2}k(u_2 - u_1)^2 - fx_1u_1 - fx_2u_2 \\ &= \frac{1}{2}k(u_2^2 - 2u_1u_2 + u_1^2) - fx_1u_1 - fx_2u_2\end{aligned} \qquad \text{Eq 41}$$

So,

$$\begin{aligned}\frac{\partial\pi_p}{\partial u_1} &= \frac{k}{2}(-2u_2 + 2u_1) - fx_1 = 0 \\ \frac{\partial\pi_p}{\partial u_2} &= \frac{k}{2}(2u_2 - 2u_1) - fx_2 = 0\end{aligned} \qquad \text{Eq 42}$$

Therefore,

$$\begin{aligned}fx_1 &= k(u_1 - u_2) \\ fx_2 &= k(-u_1 + u_2)\end{aligned} \qquad \text{Eq 43}$$

In matrix form, we obtain:

$$\begin{Bmatrix} fx_1 \\ fx_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \text{Eq 44}$$

Which is the same result as the one obtained with the displacement method (Eq 5).

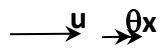
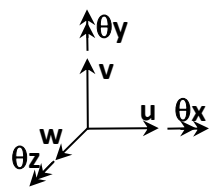
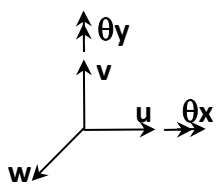
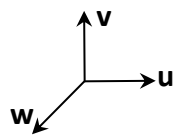
5. ELEMENT STIFFNESS MATRIX FOR VARIOUS TOPOLOGIES

The previously presented principle of minimum potential energy (see paragraph 4, page 26) is used by the FEA software to assemble the stiffness matrix of a structure. In a real problem, it becomes rapidly complex, since the structures are modeled using different topologies of elements (1D, 2D, and 3D). Before tackling the topic of stiffness matrices, let's look at the degrees of freedom and shape function concepts. It is assumed that you are already familiar with matrix algebra.

5.1. DEGREES OF FREEDOM

In FEA, the degrees of freedom (DoFs) are the fundamental variables. In mechanical structural analysis, they are usually translations and rotations at the nodes. It is important to keep in mind that not all types of degrees of freedom are necessarily transmitted by all types of elements. Degrees of freedom are fundamental to the understanding of how loads can be applied, how boundary conditions restrain the model, and how the different types of elements can be connected. A translational degree of freedom stipulates that a force can be transmitted through the nodes, while a rotational degree of freedom stipulates that a moment can be transmitted though the nodes. Table 1 below summarizes the common structural types of elements, along with their active degrees of freedom:

TABLE 1 - DOF'S FOR COMMON STRUCTURAL ELEMENTS

ELEMENT	DEGREES OF FREEDOM	
Truss	1 Translation: Tx 1 Rotation: Rx	
Beam	3 Translations: Tx Ty Tz 3 Rotations: Rx Ry Rz	
Shell	3 Translations: Tx Ty Tz 2 In-plane rotations: Rx Ry	
Solid	3 Translations: Tx Ty Tz	

Based on Table 1, you can deduce which elements you can connect or not, at the risk of creating mechanisms: two connected structural beams, for example, will behave like a fully welded connection, since the beam elements transmit three translations (one axial and two shear planes) and three rotations (one torsional moment and two bending moments). However, a beam element connected to a truss element may behave like a pinned joint because the truss element cannot transmit all rotations.

An FEA analyst must understand the concept of degrees of freedom very well to select the proper elements among all the elements in the FEA software library.

The size of the stiffness matrix of each element depends on the number of degrees of freedom associated with each node of the element. Indeed, for an element composed of n nodes having N degrees of freedom per node, the elemental stiffness matrix order is n x N. This concept of degrees of freedom number per element type is fundamental, given that it drives the size of the problem and then the computing time. The more degrees of freedom in the problem, the greater the number of equations to be solved and the more computation time it takes to find the solution.

TABLE 2 – SIZE OF THE ELEMENTAL STIFFNESS MATRICES

SIZE	ELEMENT	TYPE	ORDER	NUMBER OF NODES PER ELEMENT n	NUMBER OF DEGREES OF FREEDOM PER NODE N	MATRIX SIZE
1D	Truss	Line	Linear	2	2	4 x 4
1D	Beam	Line	Linear	2	6	12 x 12
2D	Membrane	Triangular	Linear	3	2	6 x 6
2D	Membrane	Triangular	Quadratic	6	2	12 x 12
2D	Membrane	Quadrilateral	Linear	4	2	8 x 8
2D	Membrane	Quadrilateral	Quadratic	8	2	16 x 16
2D	Shell	Triangular	Linear	3	5	15 x 15
2D	Shell	Triangular	Quadratic	6	5	30 x 30
2D	Shell	Quadrilateral	Linear	4	5	20 x 20
2D	Shell	Quadrilateral	Quadratic	8	5	40 x 40
3D	Solid	Tetrahedron	Linear	4	3	12 x 12
3D	Solid	Tetrahedron	Quadratic	10	3	30 x 30
3D	Solid	Hexahedron	Linear	8	3	24 x 24
3D	Solid	Hexahedron	Quadratic	20	3	60 x 60

When you know the number of nodes that compose your model and the topology of the elements, you can predict the total number of degrees of freedom for your problem. Given today's computing resources, a model with less than 500,000 degrees of freedom is considered a small problem, and a medium problem is one with between 500,000 and three million degrees of freedom. Large models comprise more than three million degrees of freedom.

5.2. SHAPE FUNCTIONS

The discretization of the problem into several nodes defined by the FEA analyst means that the computer will solve the problem at these particular nodes. The nodal results must also be defined in the elements which connect the nodes. For this purpose, a mathematical object called a shape function is used to represent assumed behavior for a given element. How well each assumed shape function matches the true behavior directly affects the solution's accuracy.

To represent the deformation of elements under loading, we must therefore choose a mathematical function for each element. For each element, an appropriate shape function is defined to represent the distribution of displacement within the elements, using an appropriate mathematical function. The most commonly used functions are polynomials.

Depending on the number of degrees of freedom of the elements, we derive different shape functions to express the shape of the assumed displacement fields over the domain of the elements.

It is important to choose functions that guarantee the continuity of displacements for all degrees of freedom at each node of the elements. In other words, the shape functions must prevent openings, overlaps, and jumps.

5.3. 1D TRUSS ELEMENT

5.3.1. Summary of the Displacement Method

The displacement method is used in this paragraph to derive the 1D element's stiffness matrices. Fig 9 summarizes the steps of the method:

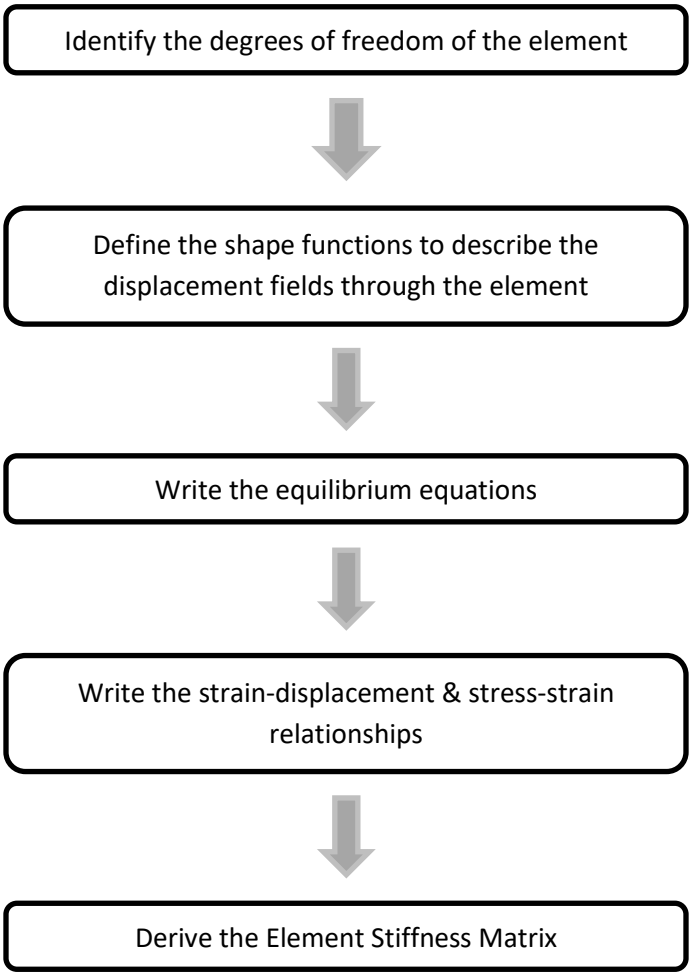


FIG 9 – DISPLACEMENT METHOD TO DERIVE THE STIFFNESS MATRICES OF ONE-DIMENSIONAL ELEMENTS

5.3.2. Truss Element in Axial Loading

Consider a simple truss element of uniform cross section A, length L and made of a homogeneous material with a Young Modulus E. The truss is subjected to an axial load.

Degrees of Freedom

The axial translations u_1 and u_2 are the only displacements at grids 1 and 2. So, this element has two degrees of freedom.

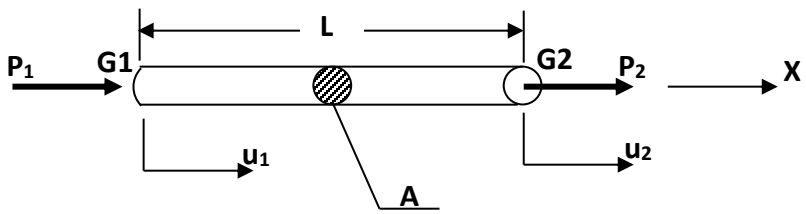


FIG 10 - TRUSS ELEMENT IN AXIAL LOADING

Shape Functions

The truss element resists axial loading only along the local x direction. A linear displacement variation along the x axis of the truss is assumed because a linear function with specified endpoints has a unique path. So, the displacement function *u* to represent the axial displacement throughout the element is written:

$$u(x) = a_1 + a_2 x \qquad \text{Eq 45}$$

The total number of coefficients a_i is equal to the total number of degrees of freedom associated with the element (here, there are two degrees of freedom, in total: one axial displacement at each of the two nodes of the element).

Let's express *u* as a function of the nodal displacements u_1 and u_2. This will permit the application of boundary conditions to nodal displacements and then relate the nodal displacements to the nodal forces. To do so, we evaluate *u* at each node and solve for a_1 and a_2.

$$u(0) = u_1 = a_1$$
$$u(L)=u_2 = u_1 + a_2 L \qquad \text{Eq 46}$$

This system gives:

$$u = \frac{u_2 - u_1}{L} x + u_1 \qquad \text{Eq 47}$$

In matrix form:

$$u = \left[1 - \frac{x}{L}, \frac{x}{L}\right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \text{Eq 48}$$

$$u = [N_1 , N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \text{Eq 49}$$

Where,

$$N_1 = 1 - \frac{x}{L}$$
$$N_2 = \frac{x}{L} \qquad \text{Eq 50}$$

are the shape functions. These shape functions Ni's express the shape of the assumed displacement over the domain of the element (x coordinate) when the ith element degree of freedom has unit value, and all other degrees of freedom are zero.

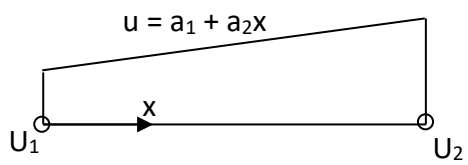


FIG 11 - DISPLACEMENT FUNCTION FOR TRUSS ELEMENT

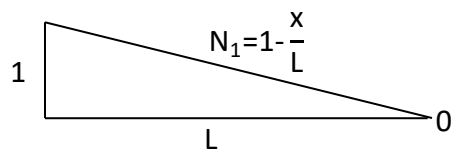


FIG 12 - SHAPE FUNCTION N_1 OVER DOMAIN OF TRUSS ELEMENT

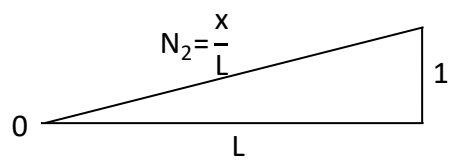


FIG 13 - SHAPE FUNCTION N_2 OVER DOMAIN OF TRUSS ELEMENT

In this case, N_1 and N_2 are linear functions that have the following behavior: N_1=1 at node 1 & N_1=0 at node 2 whereas N_2=0 at node 1 & N_2 =1 at node 2. So, for any axial coordinate along the truss element, $N_1 + N_2 = 1$.

<u>Equilibrium</u>

To satisfy the static equilibrium:

$$P_1 + P_2 = 0 \Rightarrow P_2 = -P_1$$

Strain-Displacement & Stress-Strain Relationships

The Strain/Displacement relation can be written as:

$$\varepsilon_X = \frac{du}{dx} = \frac{u_2 - u_1}{L} \qquad \text{Eq 51}$$

The Stress/Strain relation gives:

$$\sigma_x = E\varepsilon_x \qquad \text{Eq 52}$$

Element Stiffness Matrix

The force to stress relation is:

$$\sigma_{X1} = \frac{-P_1}{A}$$
$$\sigma_{X2} = \frac{P_2}{A} \qquad \text{Eq 53}$$

So,

$$-P_1 = \frac{EA}{L}u_2 - \frac{EA}{L}u_1 \qquad \text{Eq 54}$$

And

$$P_2 = \frac{EA}{L}u_2 - \frac{EA}{L}u_1 \qquad \text{Eq 55}$$

Rewrite the two equations above in matrix form:

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{bmatrix} EA/L & -EA/L \\ -EA/L & EA/L \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \text{Eq 56}$$

So, the 2 x 2 truss element axial stiffness matrix is:

$$[K]_{Truss\ Axial} = \begin{bmatrix} EA/L & -EA/L \\ -EA/L & EA/L \end{bmatrix} \qquad \text{Eq 57}$$

5.3.3. Truss Element in Torsion

Degrees of Freedom

Consider a simple truss element of uniform cross section A, length L and made of a homogeneous material with a Shear Modulus G. The truss is subjected to a torsional load T. The rotations θ_1 and θ_2 are the only displacements at grids 1 and 2. So, this element has two degrees of freedom.

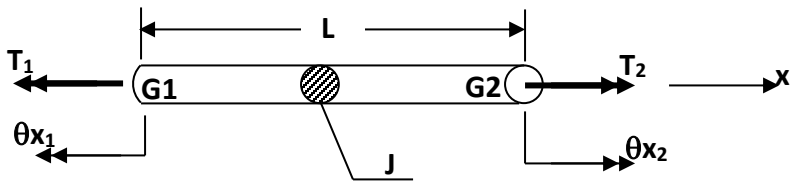


FIG 14 - TRUSS ELEMENT IN TORSION

Shape Functions

A linear angle-of-twist variation along the x axis of the truss is assumed:

$$\theta = a_1 + a_2 x \qquad \text{Eq 58}$$

So, expressing a_1 and a_2 in terms of unknown nodal angles of twist, we obtain:

$$\theta = \frac{\theta_2 - \theta_1}{L} x + \theta_1 \qquad \text{Eq 59}$$

In matrix form:

$$\theta = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$
$$u = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \text{Eq 60}$$

Where the shape functions are:

$$N_1 = 1 - \frac{x}{L}$$
$$N_2 = \frac{x}{L} \qquad \text{Eq 61}$$

Strain-Angle & Stress-Strain Relationships

The relationship between the shear strain γ and the angle of twist θ is obtained from the deformation of the bar as shown in Fig 15:

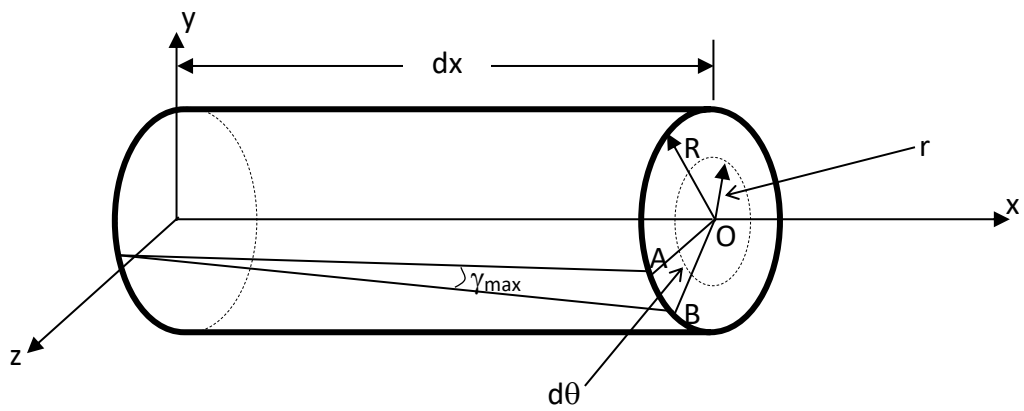


FIG 15 – SHEAR STRAIN AND ANGLE OF TWIST FOR THE TRUSS ELEMENT

Based on the assumption that all radial lines remain straight during torsional deformation, the arc length $\widehat{AB}$ is given by:

$$\widehat{AB} = \gamma_{max} dx = R d\theta$$

$$\gamma_{max} = R\frac{d\theta}{dx} \qquad \text{Eq 62}$$

So, at any radial position r, we have:

$$\gamma = r\frac{d\theta}{dx} \qquad \text{Eq 63}$$

Using the equation expressing a_1 and a_2 in terms of unknown nodal angles of twist, we have:

$$\gamma = \frac{r}{L}(\theta_2 - \theta_1) \qquad \text{Eq 64}$$

The shear stress τ is related to the shear strain γ using the shear modulus G of the material: $\tau = G\gamma$

<u>Element Stiffness Matrix</u>

We know also that the shear stress is related to the applied torque by:

$$T = \frac{\tau J}{R} \qquad \text{Eq 65}$$

Where J is the torsional constant. Then, with the nodal torque sign convention of Fig 14 we have:

$$T_2 = \frac{\tau J}{R} = \frac{G\gamma J}{R} = \frac{GRJ}{RL}(\theta_2 - \theta_1) \Rightarrow T_2 = \frac{GJ}{L}(\theta_2 - \theta_1) \qquad \text{Eq 66}$$

And,

$$T_1 = -T_2 = \frac{GJ}{L}(\theta_1 - \theta_2) \qquad \text{Eq 67}$$

In matrix format:

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{bmatrix} GJ/L & -GJ/L \\ -GJ/L & GJ/L \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} \qquad \text{Eq 68}$$

So, the 2 x 2 truss element torsional stiffness matrix is:

$$[K]_{Truss\ Torsion} = \begin{bmatrix} GJ/L & -GJ/L \\ -GJ/L & GJ/L \end{bmatrix} \qquad \text{Eq 69}$$

5.3.4. Complete Stiffness Matrix of the Truss Element

Globally, the truss element has two degrees of freedom at each node: the axial translation u_x and the rotation θx. Since the element has two nodes, the truss element has four degrees of freedom. So, we need a 4 x 4 matrix to describe the stiffness of the truss element. The complete stiffness matrix of the truss element is:

$$\begin{Bmatrix} P_1 \\ T_1 \\ P_2 \\ T_2 \end{Bmatrix} = \begin{bmatrix} EA/L & 0 & -EA/L & 0 \\ 0 & GJ/L & 0 & -GJ/L \\ -EA/L & 0 & EA/L & 0 \\ 0 & -GJ/L & 0 & GJ/L \end{bmatrix} \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix} \qquad \text{Eq 70}$$

$$[K]_{Truss} = \begin{bmatrix} EA/L & 0 & -EA/L & 0 \\ 0 & GJ/L & 0 & -GJ/L \\ -EA/L & 0 & EA/L & 0 \\ 0 & -GJ/L & 0 & GJ/L \end{bmatrix} \qquad \text{Eq 71}$$

5.3.5. Application to the Two-Truss Assembly

Now, let's consider the two-truss assembly clamped at one side and subjected to an axial load P on the other side.

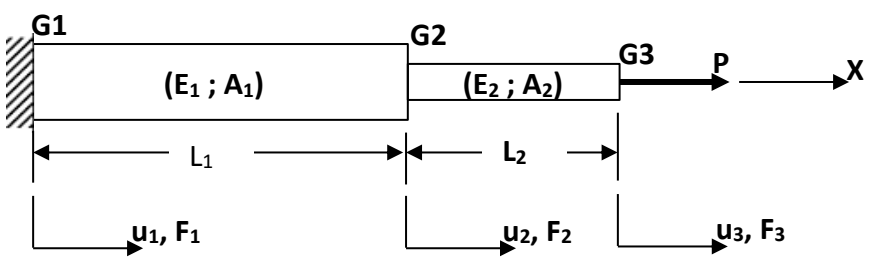


FIG 16 – THE TWO-TRUSS ASSEMBLY

Based on the previous derivation of stiffness matrix for a truss element:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = [K_1]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = [K_2]\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} \frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \qquad \text{Eq 72}$$

By assembling the two stiffness matrices by superposition, the resulting global stiffness matrix is:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} & 0 \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1} + \frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ 0 & -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \text{Eq 73}$$

By applying the external loads to the system:

$$\begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix} = \begin{bmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} & 0 \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1} + \frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ 0 & -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \text{Eq 74}$$

By imposing the boundary conditions, since the node 1 is fixed, $u_1 = 0$. We can remove row 1 and column 1 from the global stiffness matrix.

$$\begin{Bmatrix} 0 \\ P \end{Bmatrix} = \begin{bmatrix} \frac{E_1A_1}{L_1} + \frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \qquad \text{Eq 75}$$

The above system is composed of two equations with two unknowns (u_1 and u_2) and the matrix is not singular (det[K]≠0). So, the system has a solution. One way to solve the system would be to invert [K]. In practice, inverting the stiffness matrix is not an efficient method. Usually, the finite element solvers use a matrix decomposition procedure to solve the system (see paragraph 7 page 144).

5.3.6. Generic Form of the Truss Element Stiffness Matrix

To conclude this section on the truss element stiffness matrix, the generic form of the stiffness matrix using the virtual work principle is presented. The virtual work principle is an alternative to the displacement method for the derivation of an element stiffness matrix. This method is used for more complex elements such as 2D and 3D elements. However, employing the method with a simple 1D element is a good way to familiarize yourself with it.

Let's consider a generic 1D truss element subjected to an external distributed load *q(x)* (axial load per unit length). The element has a constant section *A* and length *L* and is made of a homogeneous material having a Young modulus *E*.

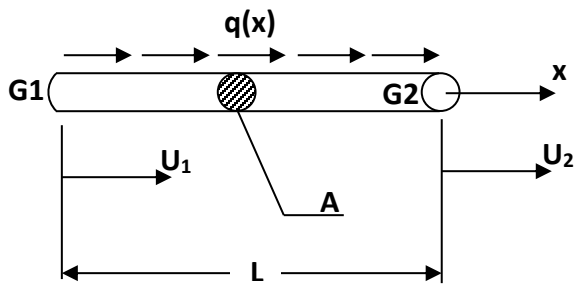


FIG 17 - 1D TRUSS ELEMENT UNDER EXTERNAL DISTRIBUTED LOAD

First, we know that the external work W_E done by the forces q(x) is stored in the truss element as internal strain energy W_I. So, let's define W_E and W_I.

<u>Internal Strain Energy W_I</u>

We know that the strain energy density at each location of the linear-elastic truss element subjected to a one-dimensional state of stress σ and strain ε is:

$$U = \frac{1}{2}\sigma\varepsilon \qquad \text{Eq 76}$$

Then, the total strain energy on the total volume V of the truss element is:

$$W_I = \int_V \frac{1}{2}\sigma\varepsilon dV \qquad \text{Eq 77}$$

Since we have σ=Eε=constant over the section A of the truss element and since dV=A.dx, the above equation can be written as:

$$W_I = \int_L \frac{1}{2}.\varepsilon.E.\varepsilon.A.dx \qquad \text{Eq 78}$$

External Work W_E

The external work of q(x) forces applied on an infinitesimal length dx at the distance x is: q(x).dx.u(x).

So, the total external work on the length L of the truss element is:

$$W_E = \int_L q(x).u(x).dx \qquad \text{Eq 79}$$

Strain

We also know that the strain and displacements are related by the equation:

$$\varepsilon(x) = \frac{d}{dx}u(x) \qquad \text{Eq 80}$$

As shown above, the variation of u(x) along the truss element is defined by its shape functions. We demonstrated that these shape functions for the truss elements are such that the variation of displacement u(x) in the truss element is linear and can be written as:

$$u(x) = N_1(x).u_1 + N_2(x).u_2 \qquad \text{Eq 81}$$

where the shape functions N_1 and N_2 are defined as:

$$N_1 = 1 - \frac{x}{L}$$
$$N_2 = \frac{x}{L} \qquad \text{Eq 82}$$

In matrix form:

$$u(x) = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \text{Eq 83}$$

$$u(x) = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \text{Eq 84}$$

$$u(x) = [N]\{u\} \qquad \text{Eq 85}$$

where

- [N] is the shape function vector
- {u} is the nodal displacement vector

So, if we come back to the strain-displacement equation:

$$\varepsilon(x) = \frac{d}{dx}u(x) = \left[\frac{d}{dx}[N]\right]\{u\} \qquad \text{Eq 86}$$

$$\varepsilon(x) = \left[-\frac{1}{L}, \frac{1}{L}\right]\{u\} \qquad \text{Eq 87}$$

where [B] is called the Strain-Displacement Matrix:

$$[B] = \left[-\frac{1}{L}, \frac{1}{L}\right] \qquad \text{Eq 88}$$

And

$$[B]^T = \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} \qquad \text{Eq 89}$$

So,

$$\varepsilon(x) = [B]\{u\} = [u][B]^T \qquad \text{Eq 90}$$

<u>The Total Potential Energy & the Virtual Work Principle</u>

The total potential energy of the truss element is given by:

$$\pi_p = W_I - W_E \qquad \text{Eq 91}$$

The virtual work principle states that the total work done by all forces acting on the truss element in static equilibrium is null for any infinitesimal virtual displacement δu:

$$\delta\pi_p = \delta W_I - \delta W_E = 0 \qquad \text{Eq 92}$$

From Eq 78 we have,

$$\delta W_I = \frac{1}{2}\int_L (\varepsilon.E.\delta\varepsilon + \delta\varepsilon.E.\varepsilon).A.dx = \int_L \varepsilon.E.\delta\varepsilon.A.dx \qquad \text{Eq 93}$$

And from Eq 79,

$$\delta W_E = \int_L q(x).\delta u(x).dx \qquad \text{Eq 94}$$

Then, as per the equations of δW_E and δW_I derived above, we have:

$$\int_L \varepsilon.E.\delta\varepsilon.A.dx = \int_L q(x).\delta u(x).dx \qquad \text{Eq 95}$$

So, using Eq 85 & Eq 90 in Eq 95, we obtain:

$$[\delta u]\int_L A.E.[B]^T.[B].\{u\}.dx = [\delta u]\int_L q_x\{N\}.dx \qquad \text{Eq 96}$$

Where [δu] is the vector of nodal virtual displacements.

Then,

$$\left[\int_L A.E.[B]^T.[B].dx\right].\{u\} = \int_L q_x.\{N\}.dx \qquad \text{Eq 97}$$

This can be written as:

$$[K]\{u\} = \{P\} \qquad \text{Eq 98}$$

Where {u} is the displacement vector and the truss element stiffness matrix [K] in generic format is:

$$[K] = \int_L A.E.[B]^T.[B].dx \qquad \text{Eq 99}$$

Since the nodal forces {F} produce the same external work as the distributed load q(x), the work-equivalent nodal force vector is:

$$\{F\} = \int_L q_x.\{N\}.dx \qquad \text{Eq 100}$$

<u>Stiffness Matrix</u>

Using the generic form of the truss element stiffness Eq 99, calculate the truss element stiffness matrix derived previously:

$$[K] = \int_L A.E.[B]^T.[B].dx \qquad \text{Eq 101}$$

$$[K] = \int_L A.E.\begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix}.\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}.dx \qquad \text{Eq 102}$$

$$[K] = \int_L A.E.\begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix}.dx \qquad \text{Eq 103}$$

$$[K] = \frac{EA}{L^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\int_L dx \qquad \text{Eq 104}$$

$$[K] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \text{Eq 105}$$

This is the same result as Eq 57.

5.4. 1D BEAM ELEMENT

The beam element is by far the most complex element in finite element analysis because it can transfer all types of loading: axial load, transverse shear, bending moments and torsion. The beam element has therefore all six degrees of freedom: three translations and three rotations.

So, I will present the development of the beam stiffness matrix step by step. First, I will present the beam stiffness matrix based on Euler-Bernoulli beam theory (bending deformation only) and then I will present the beam stiffness matrix based on Timoshenko beam theory (including transverse shear deformation). Finally, I will introduce the torsional stiffness as well as the axial stiffness to complete the stiffness matrix development.

5.4.1. Euler-Bernoulli Beam Theory

Degrees of Freedom

The basic definition of a beam is: a long structural member subjected to a transverse loading producing a significant bending effect. The bending effect is characterized by two degrees of freedom: a transverse displacement u_{iy} and a rotation θ_{iz} at each end of the beam. So, let's consider a simple beam element of:

- A uniform cross-section A
- A principal moment of inertia Iz
- A length L
- Made of a homogeneous material: Young modulus E and shear modulus G

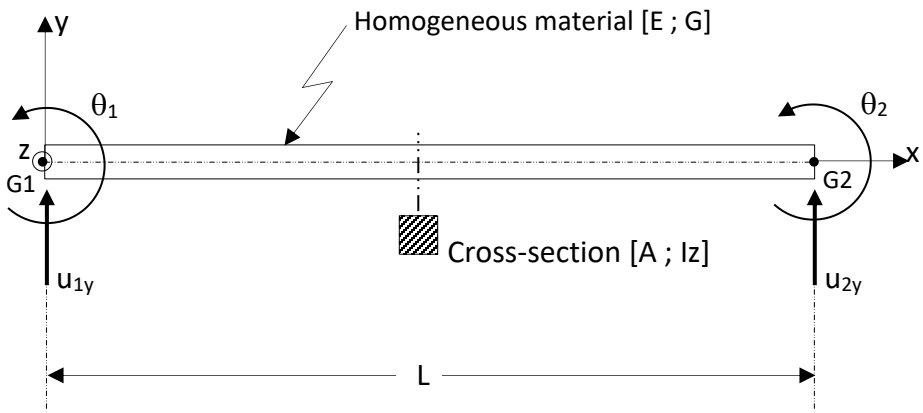


FIG 18 – BEAM ELEMENT DEGREES OF FREEDOM

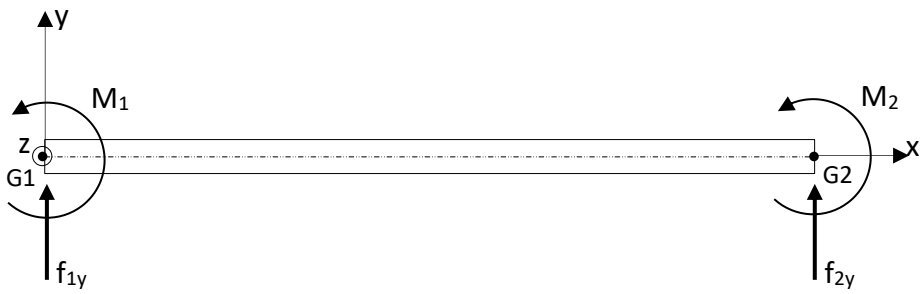


FIG 19 – BEAM ELEMENT POSITIVE FORCES AND MOMENTS

The following figure shows the sign convention used in simple beam theory for positive shear forces V and bending moments M:

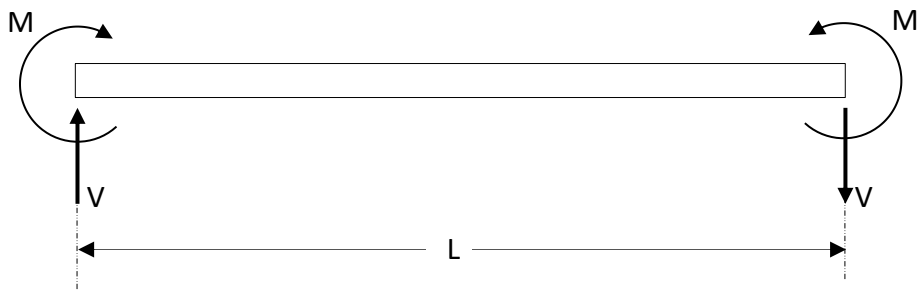


FIG 20 – BEAM THEORY SIGN CONVENTION FOR SHEAR FORCES AND BENDING MOMENTS

In the Euler-Bernoulli beam theory, the differential equation governing the elementary linear-elastic beam behavior is based on the following assumption:

The planar cross-sections perpendicular to the longitudinal axis of the beam before deformation remain plane and perpendicular after deformation.

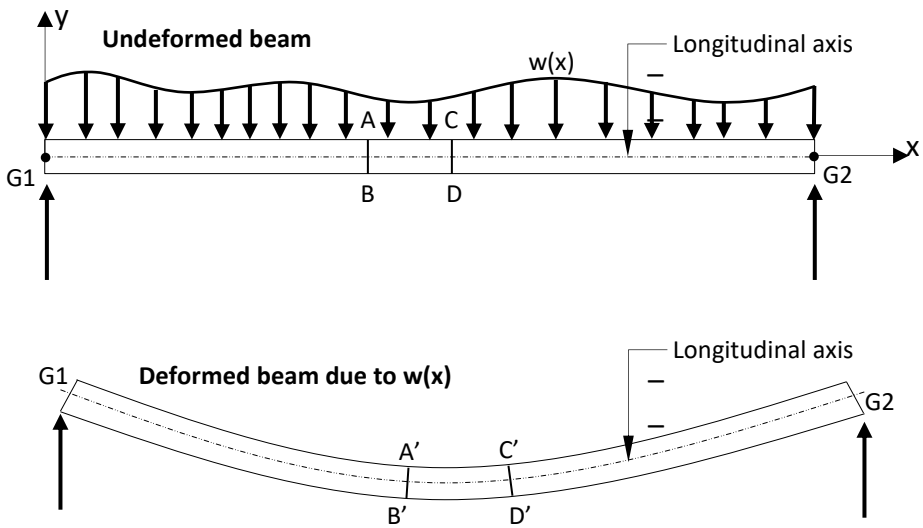


FIG 21 – BEAM DEFORMATION ACCORDING TO THE EULER-BERNOULLI THEORY

The Euler-Bernoulli assumption states that both planes A-B and A'-B' are perpendicular to the longitudinal axis. This assumption is valid only if a pure couple or a constant moment occurs in the beam.

Equilibrium

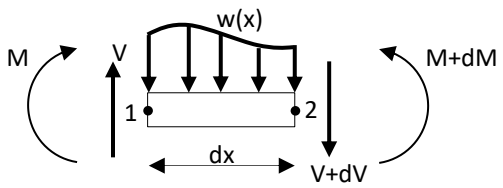


FIG 22 – EQUILIBRIUM OF THE DIFFERENTIAL BEAM ELEMENT

The equilibrium of a differential element of the beam is:

- Force equilibrium: ΣFy = 0

$$V - (V+dv) - w(x)dx = 0$$

$$-w(x)dx - dV = 0 \qquad \text{Eq 106}$$

$$w(x) = -\frac{dV}{dx}$$

- Moment equilibrium: ΣMz = 0

$$-M - Vdx + w(x)dx\left(\frac{dx}{2}\right) + M + dM = 0$$

$$-Vdx + w(x)dx\left(\frac{dx}{2}\right) + dM = 0 \qquad \text{Eq 107}$$

$$-V + w(x)\left(\frac{dx}{2}\right) + \frac{dM}{dx} = 0$$

The limit of the above equation as dx tends to 0 gives:

$$V = \frac{dM}{dx} \qquad \text{Eq 108}$$

From Eq 106 & Eq 108:

$$w(x) = -\frac{d^2M}{dx^2} \qquad \text{Eq 109}$$

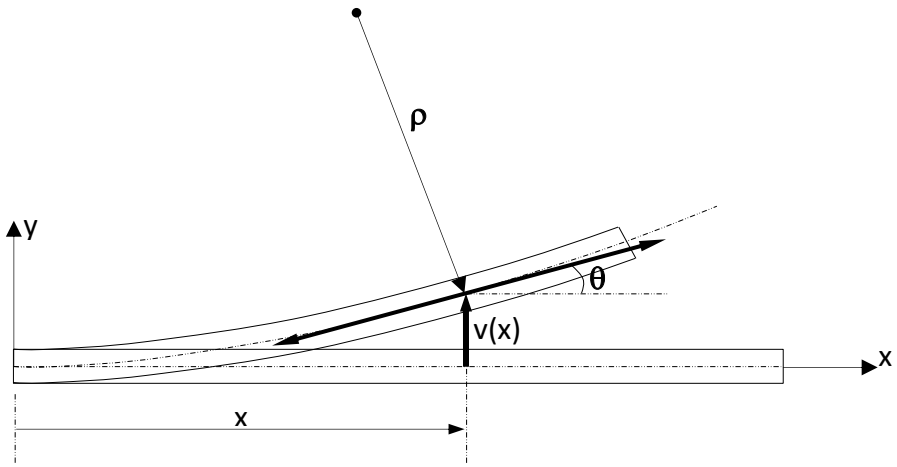


FIG 23 – BEAM CURVATURE

The curvature of the beam is related to the moment by:

$$\frac{1}{\rho} = \frac{M}{EI} \qquad \text{Eq 110}$$

For small slopes, the curvature is given by the second derivative of the transverse displacement v(x):

$$\frac{1}{\rho} = \frac{d^2v}{dx^2} \qquad \text{Eq 111}$$

So,

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \qquad \text{Eq 112}$$

Then

$$M = EI\frac{d^2v}{dx^2} \qquad \text{Eq 113}$$

Substituting Eq 113 into Eq 109:

$$w(x) = -\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) \qquad \text{Eq 114}$$

Only for nodal forces and moments, the above equation becomes (EI=constant):

$$EI\frac{d^4v}{dx^4} = 0 \qquad \text{Eq 115}$$

Shape Functions

The beam element under bending deformation has four degrees of freedom: a transverse displacement and a rotation at each node. So, the transverse displacement variation through the element length can be described by a cubic function:

$$v(x) = a_1x^3 + a_2x^2 + a_3x + a_4 \qquad \text{Eq 116}$$

This function can be expressed as a function of the nodal degrees of freedom u_{1y}, u_{2y}, θ_1 and θ_2:

$v(0) = u_{1y} = a_4$	$v(L) = u_{2y} = a_1L^3 + a_2L^2 + a_3L + a_4$
$\frac{dv(0)}{dx} = \theta_1 = a_3$	$\frac{dv(L)}{dx} = \theta_2 = 3a_1L^2 + 2a_2L + a_3$

Eq 117

So, let's resolve the above system:

$$a_4 = u_{1y}$$

$$a_3 = \theta_1$$

$$u_{2y} = a_1L^3 + a_2L^2 + a_3L + u_{1y}$$

$$\theta_2 = 3a_1L^2 + 2a_2L + a_3 = 3a_1L^2 + 2a_2L + \theta_1$$

Eq 118

So,

$$3a_1L^2 = \theta_2 - \theta_1 - 2a_2L$$

$$a_1 = \frac{\theta_2 - \theta_1 - 2a_2L}{3L^2} \qquad \text{Eq 119}$$

Substituting the above expression of a_1 in the equation of u_{2y}, we obtain:

$$a_2 = \frac{3}{L^2}(u_{2y} - u_{1y}) - \frac{1}{L}(2\theta_1 + \theta_2) \qquad \text{Eq 120}$$

Substituting the above expression of a_2 in the equation of a_1, we obtain:

$$a_1 = -\frac{2}{L^3}(u_{2y} - u_{1y}) + \frac{1}{L^2}(\theta_1 + \theta_2) \qquad \text{Eq 121}$$

Then substituting the expressions of a_1, a_2. a_3 and a_4 in Eq 116, we obtain:

$$v(x) = \left[\frac{2}{L^3}(u_{1y} - u_{2y}) + \frac{1}{L^2}(\theta_1 + \theta_2)\right]x^3 + \left[-\frac{3}{L^2}(u_{1y} - u_{2y}) - \frac{1}{L}(2\theta_1 + \theta_2)\right]x^2 + \theta_1 x + u_{1y} \quad \text{Eq 122}$$

The above equation in matrix form can be written:

$$v = [N]\{u\} \quad \text{Eq 123}$$

where,

$$\{u\} = \begin{Bmatrix} u_{1y} \\ \theta_1 \\ u_{2y} \\ \theta_2 \end{Bmatrix} \quad \text{Eq 124}$$

And,

$$[N] = [N_1\ N_2\ N_3\ N_4]$$

With the shape functions Ni for a beam element:

$$\begin{aligned} N_1 &= \frac{1}{L^3}(2x^3 - 3x^2L + L^3) \\ N_2 &= \frac{1}{L^3}(x^3L - 2x^2L^2 + xL^3) \\ N_3 &= \frac{1}{L^3}(-2x^3 + 3x^2L) \\ N_4 &= \frac{1}{L^3}(x^3L - x^2L^2) \end{aligned} \quad \text{Eq 125}$$

When evaluated at node 1, $N_1 = 1$ (with x=0) and when evaluated at node 2, $N_1 = 0$ (with x=L). On the other hand, since N_2 is associated to θ_1 in the equation v=[N]{U}, we have:

$$\frac{dN_2}{dx} = \frac{1}{L^3}(3xL - 4xL^2 + L^3) \quad \text{Eq 126}$$

When evaluated at node 1, $dN_2/dx = 1$ (with x=0). The shape functions N_3 and N_4 have the same behavior.

Strain-Displacement & Stress-Strain Relationships

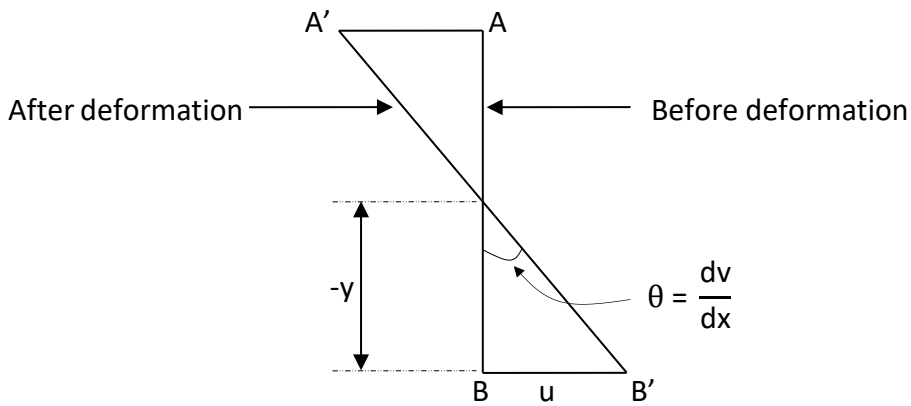


FIG 24 – ANGLE OF ROTATION OF SECTION A-B

The axial strain/displacement relationship is:

$$\varepsilon(x,y) = \frac{du}{dx} \qquad \text{Eq 127}$$

Where u is an axial displacement function. On the other hand, the transverse displacement relates to the axial displacement by:

$$u = -y\frac{dv}{dx} \qquad \text{Eq 128}$$

So, combining Eq 127 & Eq 128, we obtain:

$$\varepsilon(x,y) = -y\frac{d^2v}{dx^2} \qquad \text{Eq 129}$$

Finally, the elementary beam theory teaches us that the bending moment and the shear force are related to the transverse function by the equations:

$$M(x) = EI\frac{d^2v}{dx^2} \qquad \text{Eq 130}$$

$$V(x) = EI\frac{d^3v}{dx^3} \qquad \text{Eq 131}$$

Element Stiffness Matrix

Since the Euler-Bernoulli beam element exhibits two degrees of freedom at each node, its stiffness is described with a 4x4 matrix.

Combining Eq 122, Eq 130 & Eq 131 and using the sign conventions from Fig 19 & Fig 20, we have:

$$
\begin{aligned}
f_{1y} &= V(0) = EI\frac{d^3v(0)}{dx^3} = \frac{EI}{L^3}\left(12u_{1y} + 6L\theta_1 - 12u_{2y} + 6L\theta_2\right) \\
M_1 &= -M(0) = -EI\frac{d^2v(0)}{dx^2} = \frac{EI}{L^3}\left(6Lu_{1y} + 4L^2\theta_1 - 6Lu_{2y} + 2L^2\theta_2\right) \\
f_{2y} &= -V(L) = -EI\frac{d^3v(L)}{dx^3} = \frac{EI}{L^3}\left(-12u_{1y} - 6L\theta_1 + 12u_{2y} - 6L\theta_2\right) \\
M_2 &= M(L) = EI\frac{d^2v(L)}{dx^2} = \frac{EI}{L^3}\left(6Lu_{1y} + 2L^2\theta_1 - 6Lu_{2y} + 4L^2\theta_2\right)
\end{aligned}
\qquad \text{Eq 132}
$$

So, in matrix form, the above system becomes:

$$
\begin{Bmatrix} f_{1y} \\ M_1 \\ f_{2y} \\ M_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{1y} \\ \theta_1 \\ u_{2y} \\ \theta_2 \end{Bmatrix} \qquad \text{Eq 133}
$$

Where the stiffness matrix is:

$$
[K]_{Beam} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \qquad \text{Eq 134}
$$

The stiffness matrix $[K]_{Beam}$ relates the transverse forces and the bending moments to transverse displacements and rotations, neglecting the axial effects. This result has been obtained assuming that the beam is long and slender which means that the dimension ratio L/h (h is the depth of the beam) is large. Using the Eq 133 to predict the displacements is adequate only for these assumptions. For a short deep beam, the transverse shear deformation is usually more significant. In this case, the Timoshenko beam theory has to be considered.

5.4.2. Timoshenko Beam Theory

Equilibrium

The Timoshenko beam theory states that the plane sections do not remain perpendicular to the neutral axis of the beam after bending. Fig 25 shows a section of the differential beam dx after deformation with the cross-section no longer perpendicular to the neutral axis due to the shear force inducing an additional rotation term α.

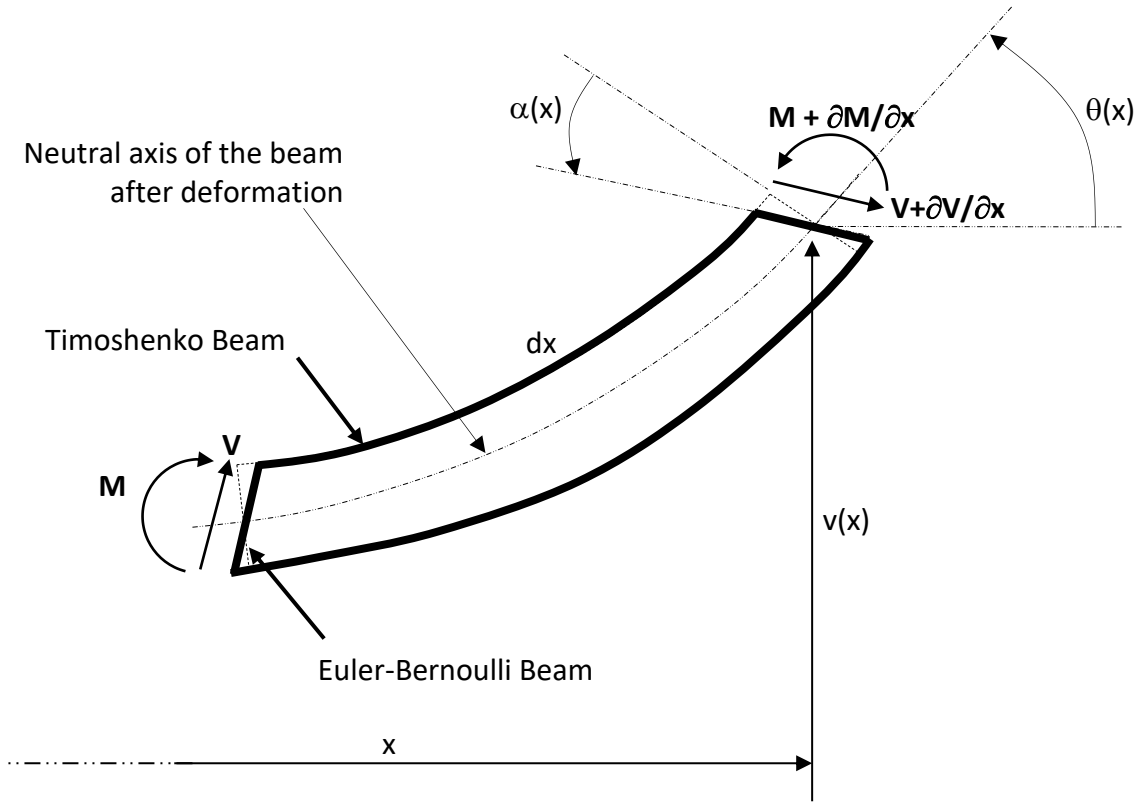


FIG 25 – TIMOSHENKO BEAM DEFORMATION

So, the total deflection of the beam at point x consists of two parts: one due to bending and one due to shear force. The slope of the deflected beam at point x is:

$$\frac{dv}{dx} = \theta(x) + \alpha(x) \qquad \text{Eq 135}$$

Where $\theta(x)$ is the rotation due to bending moment and $\alpha(x)$ is the rotation due to the transverse shear. Note that all the deflections are assumed to be small.

Strain-Displacement & Stress-Strain Relationships

The relation between the bending moment and the bending curvature is:

$$M(x) = EI\frac{d\theta(x)}{dx} \qquad \text{Eq 136}$$

The relation between the shear force and the rotation due to shear is:

$$V(x) = k_s AG\alpha(x) \qquad \text{Eq 137}$$

Where k_sA is the shear area. The shear area varies with the cross-section shapes. Refer to a mechanical handbook for the various values of k_s. Note that most of the FEA pre-processors are able to compute k_s based on the geometrical properties of the modeled cross-sections.

The shear strain γ_{yz} of the beam is represented by the difference between dv/dx and θ:

$$\gamma_{yz} = \frac{dv}{dx} - \theta = \alpha \qquad \text{Eq 138}$$

Shape Functions

The transverse shear displacement of the beam is still described by the cubic shape function Eq 116 previously defined for the Euler-Bernoulli beam:

$$v(x) = a_1x^3 + a_2x^2 + a_3x + a_4 \qquad \text{Eq 139}$$

The transverse shear strain γ is chosen consistent with the cubic polynomial for v(x), such that γ is a constant: γ=c. From Eq 106, Eq 136, Eq 137 & Eq 138:

$$V = \frac{dM}{dx} = \frac{d}{dx}\left(EI\frac{d\theta(x)}{dx}\right) = k_sAG\alpha(x) = k_sAG\gamma_{yz} = k_sAGc \qquad \text{Eq 140}$$

So,

$$c = \frac{EI}{k_sAG}\frac{d^2\theta(x)}{dx^2} \qquad \text{Eq 141}$$

From Eq 135 & Eq 116:

$$\theta(x) = \frac{dv}{dx} - \gamma = \frac{dv}{dx} - c$$

$$\theta(x) = 3a_1x^2 + 2a_2x + a_3 - c$$

$$\frac{d\theta(x)}{dx} = 6a_1x + 2a_2$$

$$\frac{d^2\theta(x)}{dx^2} = 6a_1 \qquad \text{Eq 142}$$

By combining Eq 141 & Eq 142:

$$c = \frac{6EIa_1}{k_sAG} \qquad \text{Eq 143}$$

Then, θ(x) is expressed as a polynomial in x as follow:

$$\theta(x) = 3a_1x^2 + 2a_2x + a_3 - c$$

$$\theta(x) = 3a_1x^2 + 2a_2x + a_3 - \frac{6EI}{k_sAG}a_1 \qquad \text{Eq 144}$$

With the transverse shear term β defined as:

$$\beta = \frac{EI}{k_sAG} \qquad \text{Eq 145}$$

The θ(x) equation is written:

$$\theta(x) = (3x^2 - 6\beta)\, a_1 + 2a_2x + a_3 \qquad \text{Eq 146}$$

From Eq 146 and the cubic shape function Eq 116, it is possible to compute the coefficients a_1 to a_4 in terms of nodal displacements u_{1y}, u_{2y} and rotations θ_1, θ_2 using the same approach as the one used for the Euler-Bernoulli assumption. The equations of the a_i coefficients are:

$$a_1 = \frac{2(u_{1y} - u_{2y}) + L(\theta_1 + \theta_2)}{L^3 + 12\beta L}$$

$$a_2 = \frac{3L(u_{2y} - u_{1y}) - (2L^2 + 6\beta)\theta_1 + (6\beta - L^2)\theta_2}{L^3 + 12\beta L} \qquad \text{Eq 147}$$

$$a_3 = \frac{12\beta(u_{2y} - u_{1y}) + (L^3 + 6L\beta)\theta_1 - 6\beta\theta_2}{L^3 + 12\beta L}$$

$a_4 = u_{1y}$

Substituting the a_i coefficients into the beam shape function Eq 116:

$$v(x) = \frac{2(u_{1y} - u_{2y}) + L(\theta_1 + \theta_2)}{L^3 + 12\beta L}x^3 + \frac{3L(u_{2y} - u_{1y}) - (2L^2 + 6\beta)\theta_1 + (6\beta - L^2)\theta_2}{L^3 + 12\beta L}x^2 + \frac{12\beta(u_{2y} - u_{1y}) + (L^3 + 6L\beta)\theta_1 - 6\beta\theta_2}{L^3 + 12\beta L}x + u_{1y}$$ Eq 148

<u>Element Stiffness Matrix</u>

Similarly to the Euler-Bernoulli beam stiffness matrix derivation, we have:

$$f_{1y} = V(0)$$
$$M_1 = -M(0)$$
$$f_{2y} = -V(L)$$
$$M_2 = M(L)$$ Eq 149

From Eq 131:

$$V(x) = EI\frac{d^3v}{dx^3}$$ Eq 150

From Eq 116:

$$v(x) = a_1x^3 + a_2x^2 + a_3x + a_4$$ Eq 151

So,

$$\frac{d^3v}{dx^3} = 6a_1$$ Eq 152

Then,

$$V = 6EIa_1 = V(0) = V(L)$$ Eq 153

From Eq 130:

$$M(x) = EI\frac{d^2v}{dx^2}$$ Eq 154

The second derivative of the shape function Eq 116 is:

$$\frac{d^2v}{dx^2} = 6a_1x + 2a_2$$ Eq 155

So,

$$M(0) = 2a_2EI$$
$$M(L) = EI(6a_1L+2a_2) \qquad \text{Eq 156}$$

By substituting Eq 153 & Eq 156 into Eq 149:

$$f_{1y} = V(0) = 6EIa_1$$
$$f_{2y} = -V(L) = -6EIa_1$$
$$M_1 = -M(0) = -2a_2EI \qquad \text{Eq 157}$$
$$M_2 = M(L) = EI(6a_1L+2a_2)$$

Which gives:

$$f_{1y} = EI\frac{12(u_{1y} - u_{2y}) + 6L(\theta_1 + \theta_2)}{L^3 + 12\beta L}$$
$$f_{2y} = EI\frac{12(u_{2y} - u_{1y}) - 6L(\theta_1 + \theta_2)}{L^3 + 12\beta L}$$
$$M_1 = EI\frac{6L(u_{1y} - u_{2y}) + (4L^2 + 12\beta)\theta_1 + (2L^2 - 12\beta)\theta_2}{L^3 + 12\beta L} \qquad \text{Eq 158}$$
$$M_2 = EI\frac{6L(u_{1y} - u_{2y}) + (2L^2 - 12\beta)\theta_1 + (4L^2 + 12\beta)\theta_2}{L^3 + 12\beta L}$$

In matrix form, the Eq 158 is written:

$$\begin{Bmatrix} f_{1y} \\ M_1 \\ f_{2y} \\ M_2 \end{Bmatrix} = \frac{EI}{L^3 + 12\beta L}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 + 12\beta & -6L & 2L^2 - 12\beta \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 - 12\beta & -6L & 4L^2 + 12\beta \end{bmatrix}\begin{Bmatrix} u_{1y} \\ \theta_1 \\ u_{2y} \\ \theta_2 \end{Bmatrix} \qquad \text{Eq 159}$$

Since β represents the transverse shear term, we can easily see that if we neglect the transverse shear deformation by setting β=0, we obtain the matrix of the Euler-Bernoulli beam from Eq 134.

It is usual to define a non-dimensional shear correction term φ to see the shear effect more easily:

$$\varphi = \frac{12EI}{k_s AGL^2} = \frac{12\beta}{L^2} \qquad \text{Eq 160}$$

So, the beam stiffness matrix for the Timoshenko beam is written as:

$$[K]_{Beam} = \frac{EI}{L^3(1-\varphi)} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & (4+\varphi)L^2 & -6L & (2-\varphi)L^2 \\ -12 & -6L & 12 & -6L \\ 6L & (2-\varphi)L^2 & -6L & (4+\varphi)L^2 \end{bmatrix} \qquad \text{Eq 161}$$

5.4.3. Bending and Torsion

Degrees of Freedom

We will now consider the development of the beam element stiffness matrix under bending and torsion.

The degrees of freedom at each node of the beam are:

- Vertical deflection u_{iy}
- Bending rotations θ_{iz} about the z-axis
- Torsional rotations θ_{ix} about the x-axis

So, let's consider a simple beam element of:

- A uniform cross-section A
- A principal moment of inertia Iz
- A torsional constant J
- A length L
- Made of a homogeneous material: Young modulus E and shear modulus G

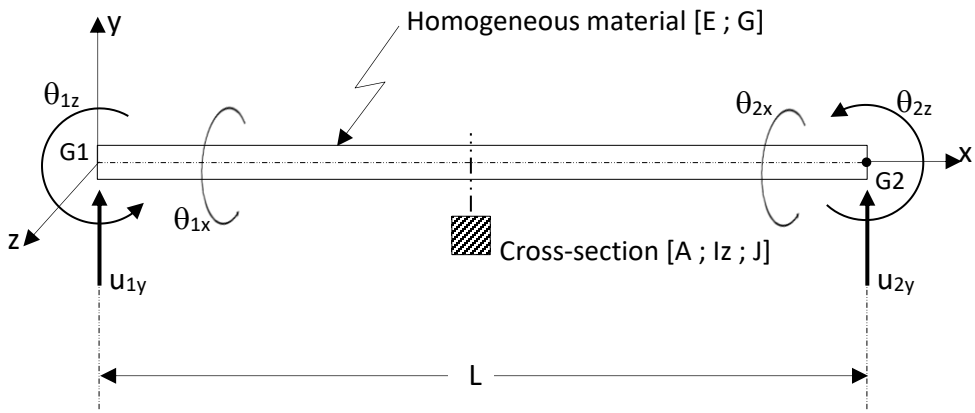


FIG 26 – BEAM ELEMENT DEGREES OF FREEDOM UNDER BENDING AND TORSION

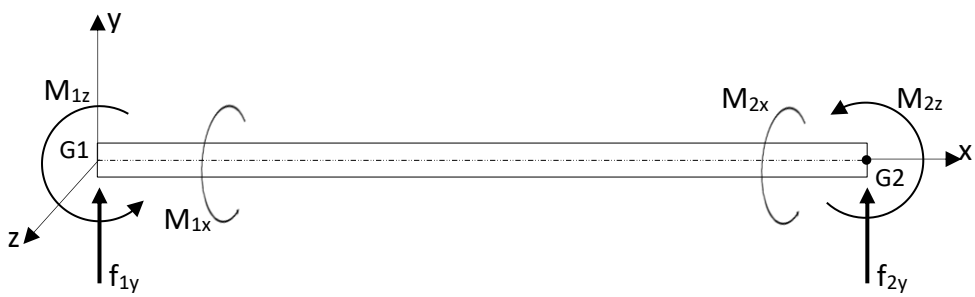


FIG 27 – BEAM ELEMENT NODAL FORCES AND MOMENTS UNDER BENDING AND TORSION

Element Stiffness Matrix

The torsional stiffness matrix of the beam element is the same as the truss element in torsion derived previously (see paragraph Truss Element in Torsion page 35):

$$\begin{Bmatrix} M_{1x} \\ M_{2x} \end{Bmatrix} = \frac{GJ}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{1x} \\ \theta_{2x} \end{Bmatrix} \qquad \text{Eq 162}$$

$$[K]_{Beam\,Torsion} = \begin{bmatrix} GJ/L & -GJ/L \\ -GJ/L & GJ/L \end{bmatrix} \qquad \text{Eq 163}$$

The torsional constant J depends on the cross-section shape. Refer to a mechanical book for the computation of J. Again, the modern FEA pre-processors are able to calculate J for every type of cross-section shape.

By combining the torsional effect of Eq 163 with the bending effect from the Euler Bernoulli beam in bending described by Eq 134, we obtain the stiffness matrix of the beam element in bending and torsion. With three degrees of freedom per node, the stiffness matrix is a 6x6:

$$\begin{Bmatrix} f_{1y} \\ M_{1x} \\ M_{1z} \\ f_{2y} \\ M_{2x} \\ M_{2z} \end{Bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} \\ 0 & \frac{GJ}{L} & 0 & 0 & -\frac{GJ}{L} & 0 \\ \frac{6EI}{L^2} & 0 & \frac{4EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} \\ 0 & -\frac{GJ}{L} & 0 & 0 & \frac{GJ}{L} & 0 \\ \frac{6EI}{L^2} & 0 & \frac{2EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} u_{1y} \\ \theta_{1x} \\ \theta_{1z} \\ u_{2y} \\ \theta_{2x} \\ \theta_{2z} \end{Bmatrix} \qquad \text{Eq 164}$$

5.4.4. Full Capabilities Beam

Degrees of Freedom

Now we can consider the full capabilities beam element stiffness matrix with six degrees of freedom:

- Axial displacements: u_{ix}
- Transverse displacements: u_{iy} & u_{iz}
- Bending rotations: θ_{iy} & θ_{iz}
- Torsional rotations: θ_{ix}

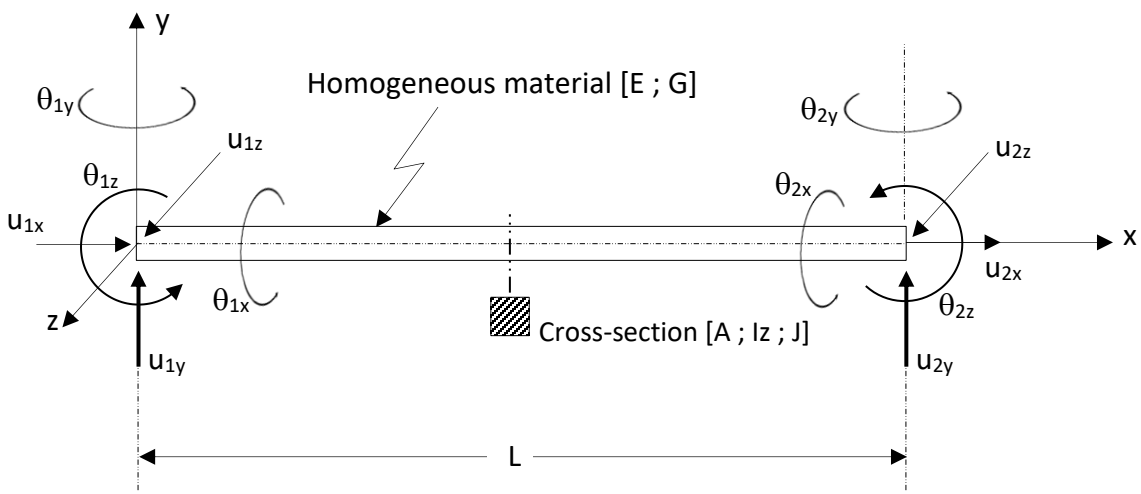


FIG 28 – FULL CAPABILITIES BEAM ELEMENT DEGREES OF FREEDOM

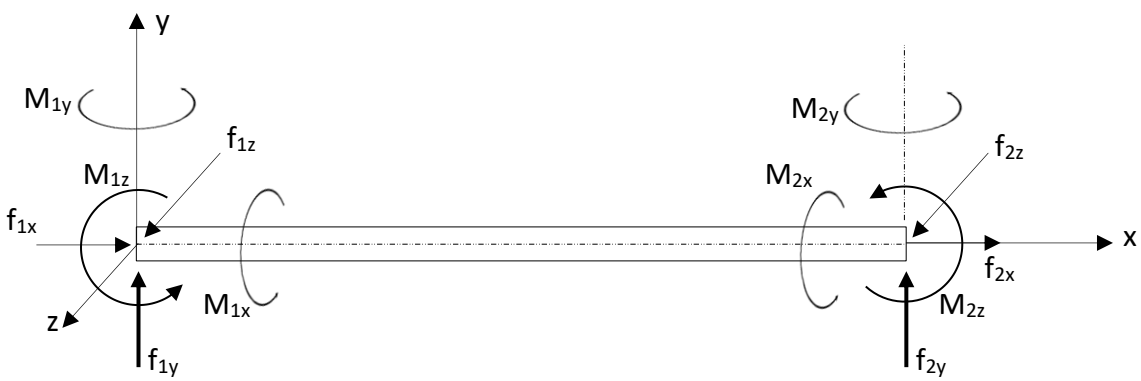


FIG 29 – FULL CAPABILITIES BEAM ELEMENT NODAL FORCES AND MOMENTS

Element Stiffness Matrix

The axial stiffness is given by Eq 57:

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} EA/L & -EA/L \\ -EA/L & EA/L \end{bmatrix} \begin{Bmatrix} u_{1x} \\ u_{2x} \end{Bmatrix} \qquad \text{Eq 165}$$

The stiffness in bending and shear in plane <x,y> is given by Eq 133:

$$\begin{Bmatrix} f_{1y} \\ M_{1z} \\ f_{2y} \\ M_{2z} \end{Bmatrix} = \frac{EI_z}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{1y} \\ \theta_{1z} \\ u_{2y} \\ \theta_{2z} \end{Bmatrix} \qquad \text{Eq 166}$$

The stiffness in bending and shear in plane <x,z> is given by Eq 133:

$$\begin{Bmatrix} f_{1z} \\ M_{1y} \\ f_{2z} \\ M_{2y} \end{Bmatrix} = \frac{EI_y}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{1z} \\ \theta_{1y} \\ u_{2z} \\ \theta_{2y} \end{Bmatrix} \qquad \text{Eq 167}$$

The torsional stiffness is given by Eq 162:

$$\begin{Bmatrix} M_{1x} \\ M_{2x} \end{Bmatrix} = \begin{bmatrix} \frac{GJ}{L} & -\frac{GJ}{L} \\ -\frac{GJ}{L} & \frac{GJ}{L} \end{bmatrix} \begin{Bmatrix} \theta_{1x} \\ \theta_{2x} \end{Bmatrix} \qquad \text{Eq 168}$$

By direct superposition of Eq 165, Eq 166, Eq 167 & Eq 168, the 12 x 12 beam element stiffness matrix with full capabilities is:

$$
[K]_{Beam} = \begin{array}{c}
\begin{array}{cccccc|cccccc}
u_{1x} & u_{1y} & u_{1z} & \theta_{1x} & \theta_{1y} & \theta_{1z} & u_{2x} & u_{2y} & u_{2z} & \theta_{2x} & \theta_{2y} & \theta_{2z}
\end{array} \\
\left[\begin{array}{cccccc|cccccc}
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
\hline
-\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
0 & 0 & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{array}\right]
\end{array}
$$

Eq 169

5.5. 2D ELEMENTS

5.5.1. Overview

The 2D elements are defined by at least three nodes up to height nodes. The elements are connected at common nodes and along common edges to form continuous structures such as those shown in Fig 30.

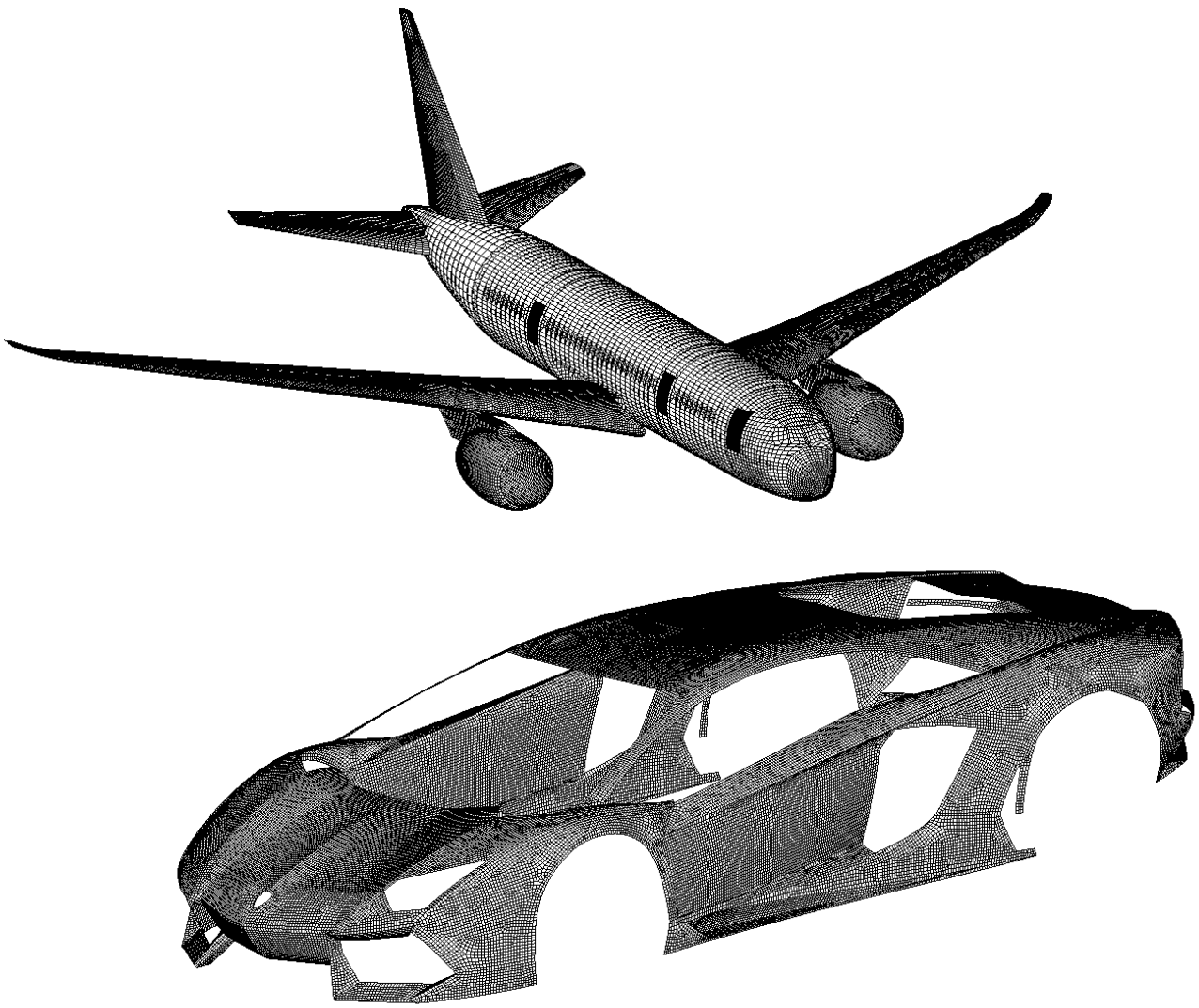


FIG 30 – 2D CONTINUOUS STRUCTURES

The derivation of the two-dimensional element stiffness matrix will be presented across the following steps:

1. *Membrane: Constant Strain Triangle (CST) element.* This 2D element is the simplest among the available 2D elements. It is called CST because it has a constant strain throughout it. The stiffness matrix will be derived using the minimum potential energy.
2. *Membrane: Linear Strain Triangle (LST) element.* This second step will illustrate the advantages of high-order elements.
3. *Plate bending element.* This third step will present the derivation of stiffness matrix for the common plate bending finite element. Two types of plates will be considered: the thin-plates described by the Kirchhoff theory and the thick-plates described by the Mindlin theory.
4. *Shell element.* Finally, the stiffness matrix of the shell element will be derived. It is a rectangular element used for dealing with the combination of membrane and bending effects.

5.5.2. Membrane: The Constant Strain Triangle (CST)

First, let's refresh your memory of the following two basic concepts:

- The concepts of plane stress and plane strain
- The concept of a two-dimensional state of stress and strain

The understanding of these concepts is very important for the development of the stiffness matrix of the 2D elements. Note that only the equations are recalled for these two concepts. For the details about their derivations, refer to a book about the elasticity theory. For the derivation of the 2D element stiffness matrix, the same pattern as the one used for the 1D elements will be followed.

<u>The Concepts of Plane Stress and Plane Strain</u>

Plane stress: it is a state of stress in which the normal stress and the shear stress perpendicular to the plane are null. In the Fig 31, the plate in the plane <x,y> subjected to the tension load P at the edge of the surface and in the plane is under a state of plane stress. The normal stress σ_Z and the shear stress τ_{XZ} and τ_{YZ} are null. A thin member, with a small *z* dimension compared to the in-plane *x* and *y* dimensions, and a member on which loads act only in the <x,y> plane can be considered to be under plane stress.

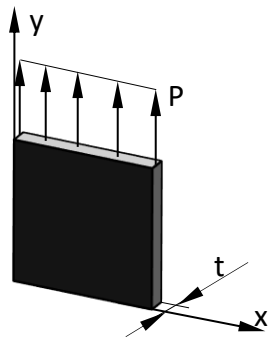


FIG 31 – PLANE STRESS

Plane strain: it is a state of strains in which the strain ε_Z normal to the <x,y> plane and the shear strains γ_{XZ} and γ_{YZ} are null. The plane strain assumption is used for long bodies, in the *z* direction, with constant cross-section, subjected to loads acting only in the plane <x,y>. The dam is a good example of plane strain state.

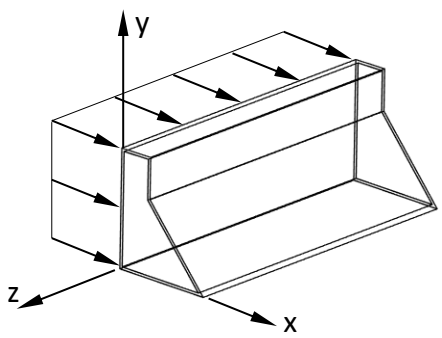


FIG 32 – PLANE STRAIN

<u>The Concept of Two-Dimensional State of Stress and Strain</u>

Let's consider the two-dimensional state of stress illustrated in Fig 33 on an infinitesimal element <dx,dy>.

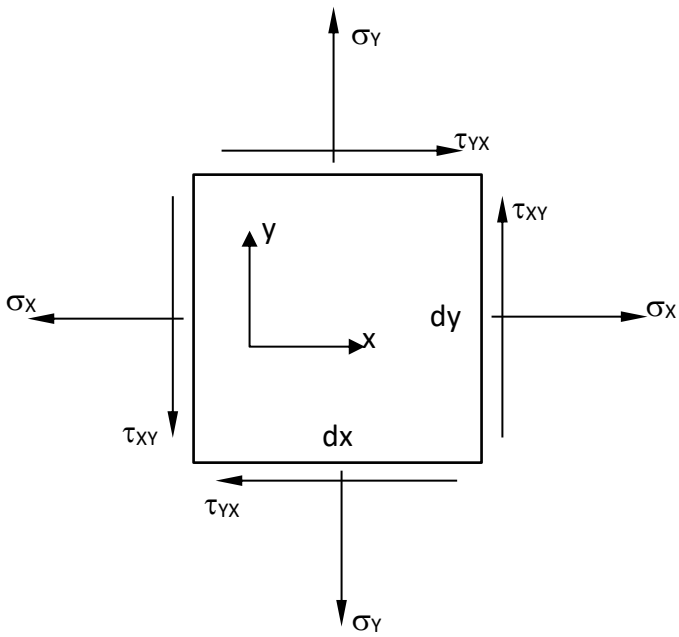


FIG 33 – TWO-DIMENSIONAL STATE OF STRESS

- dx has normal stress σ_Y acting in the x-direction
- dy has normal stress σ_x acting in the y-direction
- The shear stress τ_{XY} acts on the x edge in the y-direction
- The shear stress τ_{YX} acts on the y edge in the x-direction
- The equilibrium of the element results in $\tau_{XY}= \tau_{YX}$

This state of stress is represented by the column matrix:

$$\{\sigma\} = \begin{Bmatrix} \sigma_X \\ \sigma_Y \\ \tau_{XY} \end{Bmatrix} \qquad \text{Eq 170}$$

The displacements and rotations of the infinitesimal element <dx,dy> in the <x,y> plane is shown in Fig 34. The properties of the general two-dimensional state of strain at some point of a structure are:

- At point G, the element is displaced by an amount *u* in the *x* direction and by an amount *v* in the *y* direction.
- Along the line GH, the element extends an additional amount (∂u/∂x)dx in the *x* direction
- Along the line GJ, the element extends an additional amount (∂v/∂y)dy in the *y* direction
- Point H moves upward by (∂v/∂x)dx with respect to G
- Point J moves to the right by (∂u/∂y)dy with respect to G

So, the normal and shear strain-displacement relationships are defined as follow.

ε_X: Normal strain along the x-axis. It is the change in length per unit length of material fibers originally parallel to the x-axis when the element undergoes deformation:

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \text{Eq 171}$$

ε_Y: Normal strain along the y-axis. It is the change in length per unit length of material fibers originally parallel to the y-axis when the element undergoes deformation:

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \text{Eq 172}$$

γ_{XY}: Shear strain. It is the change of the original right angle made between dx and dy when the element undergoes deformation:

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \text{Eq 173}$$

The strains given by the three above equations are represented by the vector column matrix:

$$\{\epsilon\} = \begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \end{Bmatrix} \qquad \text{Eq 174}$$

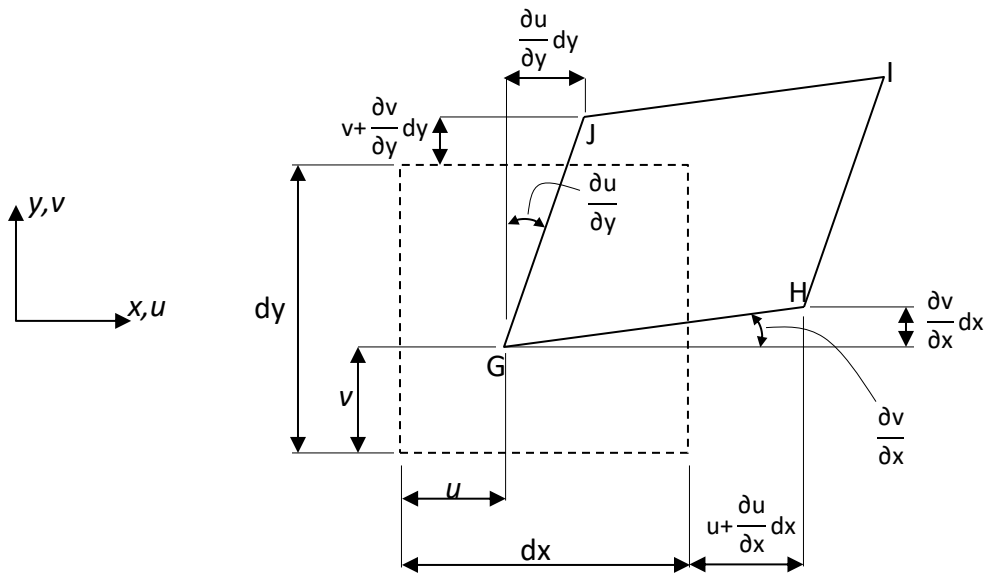


FIG 34 – DISPLACEMENTS AND ROTATIONS OF AN INFINITESIMAL ELEMENT IN THE <X,Y> PLANE

We know from the theory of elasticity that the stress-strain relationship for an isotropic body is:

$$\{\sigma\} = [D]\{\varepsilon\} \qquad \text{Eq 175}$$

Where:

- $\{\sigma\}$ and $\{\varepsilon\}$ are given by Eq 170 and Eq 174 respectively.
- [D] is the constitutive matrix.

The constitutive matrix [D] for plane stress and plane strain conditions is given by (E is the Young modulus and ν is the Poisson's ratio):

- Plane Stress: $\sigma_Z = \tau_{XZ} = \tau_{YZ} = 0$

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad \text{Eq 176}$$

- Plane Strain: $\varepsilon_Z = \gamma_{XZ} = \gamma_{YZ} = 0$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \qquad \text{Eq 177}$$

Degrees of Freedom

The basic definition of the membrane triangular element is a planar element having three nodes. Each node has two degrees of freedom representing the *x* and *y* translations of the node. Therefore, the triangular element has 3 nodes times 2 translations per node equal 6 degrees of freedom. The nodal displacements will be represented by *u* in x-direction and *v* in the y-direction.

- $(x_p ; y_p)$ are the known nodal coordinates of node p in the plane <x,y>
- $(x_q ; y_q)$ are the known nodal coordinates of node q in the plane <x,y>
- $(x_r ; y_r)$ are the known nodal coordinates of node r in the plane <x,y>
- $(u_p ; v_p)$ are the unknown degrees of freedom of node p in the plane <x,y>
- $(u_q ; v_q)$ are the unknown degrees of freedom of node q in the plane <x,y>
- $(u_r ; v_r)$ are the unknown degrees of freedom of node r in the plane <x,y>

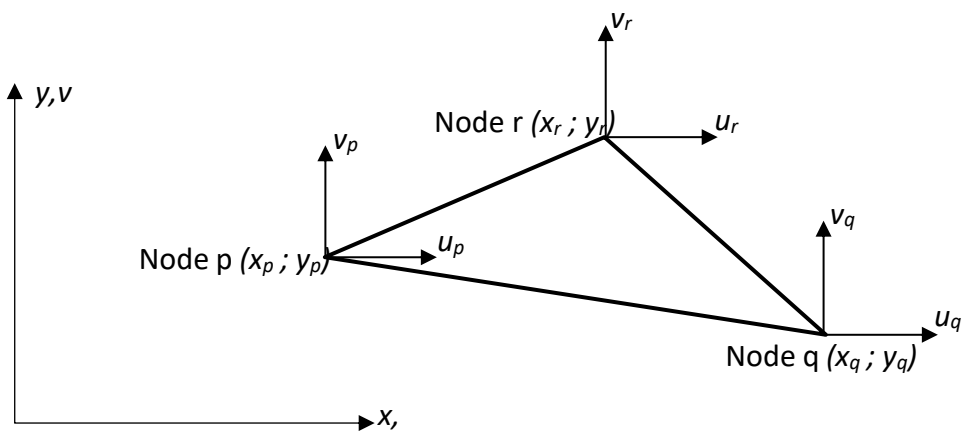


FIG 35 – NODES AND DEGREES OF FREEDOM OF THE CONSTANT STRAIN TRIANGULAR ELEMENT

The nodal displacement vector is written:

$$\{\delta\} = \begin{Bmatrix} \delta_p \\ \delta_q \\ \delta_r \end{Bmatrix} = \begin{Bmatrix} u_p \\ v_p \\ u_q \\ v_q \\ u_r \\ v_r \end{Bmatrix} \qquad \text{Eq 178}$$

Shape Functions

Since the CST element has six nodal unknowns, six constants a_i are needed to describe the complete displacement field. So, the linear displacement functions describing the displacements at any interior point $(x_i ; y_i)$ of the element are:

$$u(x,y) = a_1 + a_2x + a_3y$$
$$v(x,y) = a_4 + a_5x + a_6y \qquad \text{Eq 179}$$

These linear functions guarantee the compatibility of displacements along the edges of the elements and at the nodes shared by adjacent elements.

A general displacement function Γ is then defined from Eq 179:

$$\{\Gamma\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} \qquad \text{Eq 180}$$

To solve for the a_i's, the known coordinates of the nodal points are substituted into Eq 179:

$$\begin{aligned} u_p &= u(x_p\,;\,y_p) = a_1 + a_2x_p + a_3y_p \\ u_q &= u(x_q\,;\,y_q) = a_1 + a_2x_q + a_3y_q \\ u_r &= u(x_r\,;\,y_r) = a_1 + a_2x_r + a_3y_r \\ v_p &= v(x_p\,;\,y_p) = a_4 + a_5x_p + a_6y_p \\ v_q &= v(x_q\,;\,y_q) = a_4 + a_5x_q + a_6y_q \\ v_r &= v(x_r\,;\,y_r) = a_4 + a_5x_r + a_6y_r \end{aligned} \qquad \text{Eq 181}$$

Let's solve for the a_i's beginning with the three first equations of Eq 181 in matrix form:

$$\begin{Bmatrix} u_p \\ u_q \\ u_r \end{Bmatrix} = \begin{bmatrix} 1 & x_p & y_p \\ 1 & x_q & y_q \\ 1 & x_r & y_r \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \qquad \text{Eq 182}$$

$$\{u\} = [X]\{a\} \qquad \text{Eq 183}$$

So, we can solve for a_i's by:

$$\{a\} = [X]^{-1}\{u\} \qquad \text{Eq 184}$$

The inverse of [X] is calculated by the method of cofactors:

$$[X]^{-1} = \frac{1}{|X|} \begin{bmatrix} \alpha_p & \alpha_q & \alpha_r \\ \beta_p & \beta_q & \beta_r \\ \gamma_p & \gamma_q & \gamma_r \end{bmatrix} \qquad \text{Eq 185}$$

Where |X| is the determinant of [X]:

$$|X| = x_p(y_q - y_r) + x_q(y_r - y_p) + x_r(y_p - y_q) \qquad \text{Eq 186}$$

and

$\alpha_p = x_q y_r - y_q x_r \quad \alpha_q = y_p x_r - x_p y_r \quad \alpha_r = x_p y_q - y_p x_q$

$\beta_p = y_q - y_r \quad \beta_q = y_r - y_p \quad \beta_r = y_p - y_q$ Eq 187

$\gamma_p = x_r - x_q \quad \gamma_q = x_p - x_r \quad \gamma_r = x_q - x_p$

So, Eq 184 is written:

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{|X|} \begin{bmatrix} \alpha_p & \alpha_q & \alpha_r \\ \beta_p & \beta_q & \beta_r \\ \gamma_p & \gamma_q & \gamma_r \end{bmatrix} \begin{Bmatrix} u_p \\ u_q \\ u_r \end{Bmatrix} \qquad \text{Eq 188}$$

Similarly, the last three a_i's of Eq 181 are solved:

$$\begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{|X|} \begin{bmatrix} \alpha_p & \alpha_q & \alpha_r \\ \beta_p & \beta_q & \beta_r \\ \gamma_p & \gamma_q & \gamma_r \end{bmatrix} \begin{Bmatrix} v_p \\ v_q \\ v_r \end{Bmatrix} \qquad \text{Eq 189}$$

Expressing Eq 179 in matrix form:

$$\{u\} = [1 \quad x \quad y] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \qquad \text{Eq 190}$$

And substituting Eq 188 into Eq 190 and expanding:

$$\{u\} = [1 \quad x \quad y] \frac{1}{|X|} \begin{bmatrix} \alpha_p & \alpha_q & \alpha_r \\ \beta_p & \beta_q & \beta_r \\ \gamma_p & \gamma_q & \gamma_r \end{bmatrix} \begin{Bmatrix} u_p \\ u_q \\ u_r \end{Bmatrix} \qquad \text{Eq 191}$$

$$\{u\} = \frac{1}{|X|} [1 \quad x \quad y] \begin{Bmatrix} \alpha_p u_p + \alpha_q u_q + \alpha_r u_r \\ \beta_p u_p + \beta_q u_q + \beta_r u_r \\ \gamma_p u_p + \gamma_q u_q + \gamma_r u_r \end{Bmatrix} \qquad \text{Eq 192}$$

Eq 192 gives the *u(x,y)* displacement function:

$$u(x,y) = \frac{1}{|X|} \{ (\alpha_p + \beta_p x + \gamma_p y) u_p + (\alpha_q + \beta_q x + \gamma_q y) u_q + (\alpha_r + \beta_r x + \gamma_r y) u_r \} \qquad \text{Eq 193}$$

Similarly, replacing the *u* degree of freedom by the *v* degree of freedom in Eq 193, we obtain the *v(x,y)* displacement function:

$$v(x,y) = \frac{1}{|X|}\{(\alpha_p + \beta_p x + \gamma_p y)v_p + (\alpha_q + \beta_q x + \gamma_q y)v_q + (\alpha_r + \beta_r x + \gamma_r y)v_r\} \quad \text{Eq 194}$$

From Eq 193 & Eq 194, the following shape functions are considered:

$$N_p = \frac{1}{|X|}(\alpha_p + \beta_p x + \gamma_p y)$$
$$N_q = \frac{1}{|X|}(\alpha_q + \beta_q x + \gamma_q y) \quad \text{Eq 195}$$
$$N_r = \frac{1}{|X|}(\alpha_r + \beta_r x + \gamma_r y)$$

Using the shapes functions Eq 195 into Eq 193 & Eq 194, the following equations are obtained:

$$u(x,y) = N_p u_p + N_q u_q + N_r u_r$$
$$v(x,y) = N_p v_p + N_q v_q + N_r v_r \quad \text{Eq 196}$$

The general displacement function Γ is then written as:

$$\{\Gamma\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{Bmatrix} N_p u_p + N_q u_q + N_r u_r \\ N_p v_p + N_q v_q + N_r v_r \end{Bmatrix} \quad \text{Eq 197}$$

$$\{\Gamma\} = \begin{bmatrix} N_p & 0 & N_q & 0 & N_r & 0 \\ 0 & N_p & 0 & N_q & 0 & N_r \end{bmatrix} \begin{Bmatrix} u_p \\ v_p \\ u_q \\ v_q \\ u_r \\ v_r \end{Bmatrix} \quad \text{Eq 198}$$

So,

$$\{\Gamma\} = [N]\{\delta\} \quad \text{Eq 199}$$

Where,

$$\{N\} = \begin{bmatrix} N_p & 0 & N_q & 0 & N_r & 0 \\ 0 & N_p & 0 & N_q & 0 & N_r \end{bmatrix} \quad \text{Eq 200}$$

The shape functions N_p, N_q & N_r represent the shape of the general displacement function Γ over the surface of the element.

Strain-Displacement & Stress-Strain Relationships

Let's express the element strains and stresses in terms of the unknown nodal displacements. It is known that the strains of the two-dimensional element are:

$$\{\epsilon\} = \begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \qquad \text{Eq 201}$$

From Eq 196:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(N_p u_p \ + \ N_q u_q \ + \ N_r u_r\right) \qquad \text{Eq 202}$$

However, in the CST element it is assumed that:

- $u_p = u(x_p,y_p)$ = constant
- $u_q = u(x_q,y_q)$ = constant
- $u_r = u(x_r,y_r)$ = constant

The derivatives of u_p, u_q and u_r with respect of the variable *x* are null, so:

$$\frac{\partial u}{\partial x} \ = \ \frac{\partial N_p}{\partial x} u_p \ + \ \frac{\partial N_q}{\partial x} u_q \ + \ \frac{\partial N_r}{\partial x} u_r \qquad \text{Eq 203}$$

From Eq 195, the derivatives of the shape functions are:

$$\frac{\partial N_p}{\partial x} \ = \ \frac{1}{|X|}\frac{\partial}{\partial x}\left(\alpha_p \ + \ \beta_p x \ + \ \gamma_p y\right) \ = \frac{\beta_p}{|X|} \qquad \text{Eq 204}$$

$$\frac{\partial N_q}{\partial x} \ = \ \frac{\beta_q}{|X|} \qquad \text{Eq 205}$$

$$\frac{\partial N_r}{\partial x} \ = \ \frac{\beta_r}{|X|} \qquad \text{Eq 206}$$

Substituting Eq 204, Eq 205 & Eq 206 into Eq 203:

$$\frac{\partial u}{\partial x} \ = \ \frac{1}{|X|}\left(\beta_p u_p \ + \ \beta_q u_q \ + \ \beta_r u_r\right) \qquad \text{Eq 207}$$

Similarly:

$$\frac{\partial v}{\partial y} \ = \ \frac{1}{|X|}\left(\gamma_p v_p \ + \ \gamma_q v_q \ + \ \gamma_r v_r\right) \qquad \text{Eq 208}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{|X|}\left(\gamma_p u_p + \beta_p v_p + \gamma_q u_q + \beta_q v_q + \gamma_r u_r + \beta_r v_r\right) \qquad \text{Eq 209}$$

Using Eq 178, Eq 207, Eq 208 & Eq 209 into Eq 201:

$$\{\epsilon\} = \frac{1}{|X|}\begin{bmatrix} \beta_p & 0 & \beta_q & 0 & \beta_r & 0 \\ 0 & \gamma_p & 0 & \gamma_q & 0 & \gamma_r \\ \gamma_p & \beta_p & \gamma_q & \beta_q & \gamma_r & \beta_r \end{bmatrix}\begin{Bmatrix} u_p \\ v_p \\ u_q \\ v_q \\ u_r \\ v_r \end{Bmatrix} \qquad \text{Eq 210}$$

$$\{\epsilon\} = \begin{bmatrix} [B_p] & [B_q] & [B_r] \end{bmatrix}\begin{Bmatrix} \delta_p \\ \delta_q \\ \delta_r \end{Bmatrix} \qquad \text{Eq 211}$$

$$\{\epsilon\} = [B]\{\delta\} \qquad \text{Eq 212}$$

Where [B] is:

$$[B] = \begin{bmatrix} [B_p] & [B_q] & [B_r] \end{bmatrix} \qquad \text{Eq 213}$$

[B] is a matrix independent of the x and y coordinates. It depends only on the element nodal coordinates.

$$[B_p] = \frac{1}{|X|}\begin{bmatrix} \beta_p & 0 \\ 0 & \gamma_p \\ \gamma_p & \beta_p \end{bmatrix}$$

$$[B_q] = \frac{1}{|X|}\begin{bmatrix} \beta_q & 0 \\ 0 & \gamma_q \\ \gamma_q & \beta_q \end{bmatrix} \qquad \text{Eq 214}$$

$$[B_r] = \frac{1}{|X|}\begin{bmatrix} \beta_r & 0 \\ 0 & \gamma_r \\ \gamma_r & \beta_r \end{bmatrix}$$

From Eq 175:

$$\{\sigma\} = [D]\{\varepsilon\}$$

Where [D] is given by Eq 176 for a plane stress problem and by Eq 177 for a plane strain problem.

Using Eq 175 and Eq 212, the stress in the element is:

$$\{\sigma\} = [D][B]\{\delta\} \qquad \text{Eq 215}$$

Note that $\{\sigma\}$ is constant in the CST element.

Element Stiffness Matrix

The principle of minimum potential energy is used to derive the stiffness matrix of the CST element. The total potential energy of the element is given by:

$$\pi_p = U + \Omega_b + \Omega_p + \Omega_s \qquad \text{Eq 216}$$

Where,

- U is the strain energy
- Ω_b is the potential energy of the body forces
- Ω_p is the potential energy of the concentrated forces
- Ω_s is the potential energy of distributed loads

The strain energy U is given by:

$$U = \frac{1}{2}\iiint_V \{\varepsilon\}^T\{\sigma\}\, dV \qquad \text{Eq 217}$$

Using Eq 175:

$$U = \frac{1}{2}\iiint_V \{\varepsilon\}^T[D]\{\varepsilon\}\, dV \qquad \text{Eq 218}$$

Substituting Eq 212 into Eq 218:

$$U = \frac{1}{2}\iiint_V [B]^T\{\delta\}^T[D][B]\{\delta\}\, dV \qquad \text{Eq 219}$$

The potential energy Ω_b of the body forces is given by:

$$\Omega_b = -\iiint_V \{\Gamma\}^T\{\Phi\} dV \qquad \text{Eq 220}$$

Where,

- $\{\Gamma\}$ is the general displacement function defined by Eq 199.
- $\{\Phi\}$ is the body weight density matrix in units of $[F]/[L^3]$

Substituting Eq 199 into Eq 220:

$$\Omega_b = -\iiint_V [N]^T\{\delta\}^T\{\Phi\}dV \qquad \text{Eq 221}$$

The potential energy Ω_p of the concentrated forces is given by:

$$\Omega_p = -\{\delta\}^T\{P\} \qquad \text{Eq 222}$$

Where,

- $\{\delta\}$ is the nodal displacement vector
- $\{P\}$ is the concentrated external loads vector

The potential energy Ω_s of the distributed loads is given by:

$$\Omega_s = -\iint_S \{\Gamma_s\}^T\{Q_s\}dS \qquad \text{Eq 223}$$

Where,

- $\{\Gamma_s\}$ is the field of surface displacements through which the distributed loads act
- $\{Q_s\}$ is the distributed loads in units od $[F]/[L^2]$
- S is the surface over which the distributed loads $\{Q_s\}$ act

Similar to Eq 199, we can express $\{\Gamma_s\}$ in relation with the shape function matrix $[N_s]$ evaluated along the surface where the distributed loads act:

$$\{\Gamma_s\} = [N_s]\{\delta\} \qquad \text{Eq 224}$$

So, Eq 223 becomes:

$$\Omega_s = -\iint_S [N_s]^T\{\delta\}^T\{Q_s\}dS \qquad \text{Eq 225}$$

Then, substituting Eq 219, Eq 221, Eq 222 & Eq 225 into Eq 216:

$$\pi_p = \frac{1}{2}\iiint_V [B]^T\{\delta\}^T[D][B]\{\delta\}\,dV - \iiint_V [N]^T\{\delta\}^T\{\Phi\}dV - \{\delta\}T\{P\} - \iint_S [N_s]^T\{\delta\}^T\{Q_s\}dS \qquad \text{Eq 226}$$

Since the nodal displacement vector $\{\delta\}$ is not dependent of the *x* and *y* coordinates, $\{\delta\}^T$ can be taken out of the integrals:

$$\pi_p = \frac{1}{2}\{\delta\}^T\iiint_V [B]^T[D][B]\{\delta\}\,dV - \{\delta\}^T\iiint_V [N]^T\{\Phi\}dV - \{\delta\}T\{P\} - \{\delta\}^T\iint_S [N_s]^T\{Q_s\}dS \qquad \text{Eq 227}$$

The total load {f} applied on the element is the summation of the body force loads, the concentrated forces and the distributed forces used to compute the potential energies from Eq 221, Eq 222 & Eq 225.

So, the total load vector applied on the element is:

$$\{f\} = \iiint_V [N]^T\{\Phi\}dV \ + \ \{P\} \ + \ \iint_S [N_s]^T\{Q_s\}dS \qquad \text{Eq 228}$$

Substituting Eq 228 into Eq 227:

$$\pi_p \ = \ \frac{1}{2}\{\delta\}^T \iiint_V [B]^T[D][B]\{\delta\}\, dV \ - \ \{\delta\}^T\{f\} \qquad \text{Eq 229}$$

Statically, finding the equilibrium of the element corresponds to find a value of the nodal displacement vector {δ} yielding a stationary value of the potential energy π_p. This condition is obtained for:

$$\frac{\partial \pi_p}{\partial(\delta)} = 0 \qquad \text{Eq 230}$$

Computing the derivative of Eq 229 with respect to the nodal displacement vector {δ} and substituting into Eq 230:

$$\frac{\partial \pi_p}{\partial(\delta)} = \left[\iiint_V [B]^T[D][B]\, dV\right]\{\delta\} \ - \ \{f\} = 0 \qquad \text{Eq 231}$$

$$\left[\iiint_V [B]^T[D][B]\, dV\right]\{\delta\} \ = \ \{f\} \qquad \text{Eq 232}$$

From Eq 232, the stiffness matrix appears:

$$[K] = \iiint_V [B]^T[D][B]\, dV \qquad \text{Eq 233}$$

For an element with constant thickness t, Eq 233 becomes:

$$[K] = t \iint_S [B]^T[D][B]dxdy \qquad \text{Eq 234}$$

However, for the constant strain triangular element, all the terms of the integral are not a function of *x* or *y*. Then:

$$[K] = tA[B]^T[D][B] \qquad \text{Eq 235}$$

Where,

- A is the area of the triangle and is given by: A = |X|/2
- [B] is given by Eq 213
- [D] is given by Eq 176 for a plane stress problem and by Eq 177 for a plane strain problem.

The stiffness matrix [K] is a function of:

- The nodal coordinates via [B] and A
- The mechanical properties E and ν via [D]

Finally, the stiffness matrix of the constant strain triangular element is a 6 x 6 matrix written as:

$$[K] = \begin{bmatrix} [k_{pp}] & [k_{pq}] & [k_{pr}] \\ [k_{qp}] & [k_{qq}] & [k_{qr}] \\ [k_{rp}] & [k_{rq}] & [k_{rr}] \end{bmatrix} \qquad \text{Eq 236}$$

Where each submatrice is given by:

For i = p, q or r
and j = p, q or r

$$[k_{ij}] = \frac{t|X|}{2}[B_i]^T[D][B_j] \qquad \text{Eq 237}$$

Where $[B_i]$ and $[B_j]$ are defined by Eq 214. Note that each submatrice $[k_{ij}]$ is a 2 x 2 matrix.

So, the 6 x 6 matrix [k] of the CST element is:

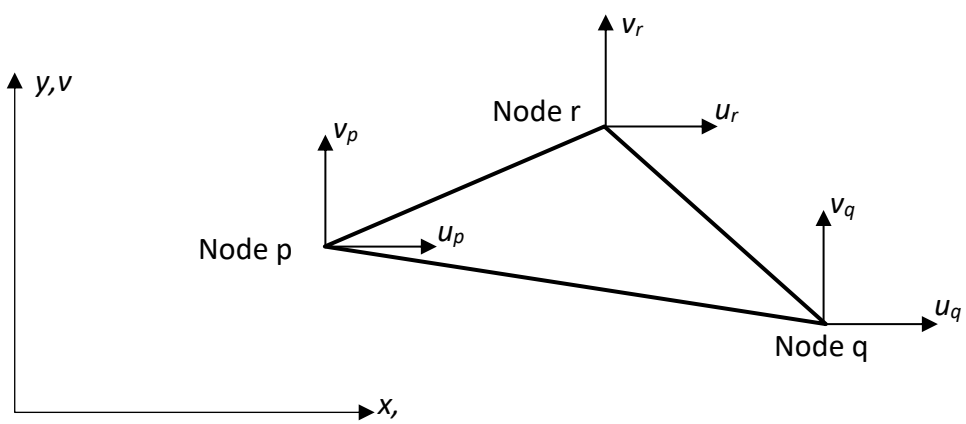


FIG 36 – THE CST ELEMENT'S COORDINATE SYSTEM AND DOF`S

$$[k] = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} \text{Node p} & \text{Node q} & \text{Node r} \\ \begin{array}{cc} u_p & v_p \end{array} & \begin{array}{cc} u_q & v_q \end{array} & \begin{array}{cc} u_r & v_r \end{array} \end{array} \\ \begin{bmatrix} \begin{bmatrix} k_{ppx} & k_{ppxy} \\ k_{ppyx} & k_{ppy} \end{bmatrix} & \begin{bmatrix} k_{pqx} & k_{pqxy} \\ k_{pqyx} & k_{pqy} \end{bmatrix} & \begin{bmatrix} k_{prx} & k_{prxy} \\ k_{pryx} & k_{pry} \end{bmatrix} \\ \begin{bmatrix} k_{qpx} & k_{qpxy} \\ k_{qpyx} & k_{qpy} \end{bmatrix} & \begin{bmatrix} k_{qqx} & k_{qqxy} \\ k_{qqyx} & k_{qqy} \end{bmatrix} & \begin{bmatrix} k_{qrx} & k_{qrxy} \\ k_{qryx} & k_{qry} \end{bmatrix} \\ \begin{bmatrix} k_{rpx} & k_{rpxy} \\ k_{rpyx} & k_{rpy} \end{bmatrix} & \begin{bmatrix} k_{rqx} & k_{rqxy} \\ k_{rqyx} & k_{rqy} \end{bmatrix} & \begin{bmatrix} k_{rrx} & k_{rrxy} \\ k_{rryx} & k_{rry} \end{bmatrix} \end{bmatrix} \end{array} \begin{array}{c} \\ u_p \\ v_p \\ u_q \\ v_q \\ u_r \\ v_r \end{array} \qquad \text{Eq 238}$$

<u>Matrix Assembly</u>

The direct stiffness method can be used to obtain the global structure stiffness matrix [K] for a structure meshed with N CST elements:

$$[K] = \sum_{e=1}^{N} [k]_e \qquad \text{Eq 239}$$

Where $[k]_e$ is the stiffness matrix of each element of the discretization. Each $[k]_e$ is computed with Eq 236.

<u>Explicit Expression of the CST Stiffness Matrix</u>

In this last paragraph about the CST element the 6 x 6 stiffness matrix is explicitly written using the Eq 235.

$$[K] \;=\; tA[B]^T[D][B] \qquad \text{Eq 240}$$

Where,

- A is the area of the triangle element
- t is the constant thickness of the element
- [D] is the constitutive matrix given by Eq 177
- [B] is given by Eq 213

We will write the CST element stiffness matrix considering the plane strain assumption. By substituting the matrices [D] and [B] into Eq 240, we have:

$$[K]=\frac{\mathrm{tE}}{4\mathrm{A}(1+\nu)(1\text{-}2\nu)}\begin{bmatrix} \beta_p & 0 & \gamma_p \\ 0 & \gamma_p & \beta_p \\ \beta_q & 0 & \gamma_q \\ 0 & \gamma_q & \beta_q \\ \beta_r & 0 & \gamma_r \\ 0 & \gamma_r & \beta_r \end{bmatrix}\begin{bmatrix} 1\text{-}\nu & \nu & 0 \\ \nu & 1\text{-}\nu & 0 \\ 0 & 0 & \frac{1\text{-}2\nu}{2} \end{bmatrix}\begin{bmatrix} \beta_p & 0 & \beta_q & 0 & \beta_r & 0 \\ 0 & \gamma_p & 0 & \gamma_q & 0 & \gamma_r \\ \gamma_p & \beta_p & \gamma_q & \beta_q & \gamma_r & \beta_r \end{bmatrix} \qquad \text{Eq 241}$$

The explicit constant strain triangle element stiffness matrix for the plane strain assumption is obtained by multiplying the matrices in Eq 241.

The stiffness matrix [K] is a function of:

- the *x* and *y* nodal coordinates thanks to the β's and γ's
- The material properties *E* and ν
- The thickness *t* of the element
- The area *A* of the element

Note that the stiffness matrix is transformed for the plane stress assumption by replacing the following terms in Eq 241:

- Replace $(1 - \nu)$ by 1
- Replace $(1 - 2\nu)/2$ by $(1 - \nu)/2$
- Replace $(1 + \nu)(1 - 2\nu)$ by $(1 - \nu^2)$

$$[K] = \frac{tE}{4A(1+\nu)(1-2\nu)}$$

$$X\begin{bmatrix} \beta_p^2(1-\nu)+\gamma_p^2\left(\frac{1-2\nu}{2}\right) & \beta_p\gamma_p\nu+\beta_p\gamma_p\left(\frac{1-2\nu}{2}\right) & \beta_p\beta_q(1-\nu)+\gamma_p\gamma_q\left(\frac{1-2\nu}{2}\right) & \beta_p\gamma_q\nu+\beta_q\gamma_p\left(\frac{1-2\nu}{2}\right) & \beta_p\beta_r(1-\nu)+\gamma_p\gamma_r\left(\frac{1-2\nu}{2}\right) & \beta_p\gamma_r\nu+\beta_r\gamma_p\left(\frac{1-2\nu}{2}\right) \\ & \gamma_p^2(1-\nu)+\beta_p^2\left(\frac{1-2\nu}{2}\right) & \beta_q\gamma_p\nu+\beta_p\gamma_q\left(\frac{1-2\nu}{2}\right) & \gamma_p\gamma_q(1-\nu)+\beta_p\beta_q\left(\frac{1-2\nu}{2}\right) & \beta_r\gamma_p\nu+\beta_p\gamma_r\left(\frac{1-2\nu}{2}\right) & \gamma_p\gamma_r(1-\nu)+\beta_p\beta_r\left(\frac{1-2\nu}{2}\right) \\ & & \beta_q^2(1-\nu)+\gamma_q^2\left(\frac{1-2\nu}{2}\right) & \beta_q\gamma_q\nu+\beta_q\gamma_q\left(\frac{1-2\nu}{2}\right) & \beta_q\beta_r(1-\nu)+\gamma_q\gamma_r\left(\frac{1-2\nu}{2}\right) & \beta_q\gamma_r\nu+\beta_r\gamma_q\left(\frac{1-2\nu}{2}\right) \\ & & & \gamma_q^2(1-\nu)+\beta_q^2\left(\frac{1-2\nu}{2}\right) & \beta_r\gamma_q\nu+\beta_q\gamma_r\left(\frac{1-2\nu}{2}\right) & \gamma_q\gamma_r(1-\nu)+\beta_q\beta_r\left(\frac{1-2\nu}{2}\right) \\ & & & & \beta_r^2(1-\nu)+\gamma_r^2\left(\frac{1-2\nu}{2}\right) & \beta_r\gamma_r\nu+\beta_r\gamma_r\left(\frac{1-2\nu}{2}\right) \\ & & & & & \gamma_r^2(1-\nu)+\beta_r^2\left(\frac{1-2\nu}{2}\right) \end{bmatrix}$$

$$\begin{array}{cccccc} S & & Y & & M & \end{array}$$

Eq 242

5.5.3. Membrane: The Linear Strain Triangle (LST)

Let's now consider the stiffness matrix computation for the high-order triangular element called the Linear Strain Triangle (LST). The same approach as the CST element is used for the development of the stiffness matrix. However, since the number of equations is double, the hand calculations are heavy. So, we will not solve the equations for the LST element, we will just expose the equations that will be solved numerically by a computer.

Degrees of Freedom

The LST element has six nodes with two degrees of freedom per node, giving a total of twelve degrees of freedom for the element. In addition to the three corner nodes, three mid-nodes are added on each side of the triangular element.

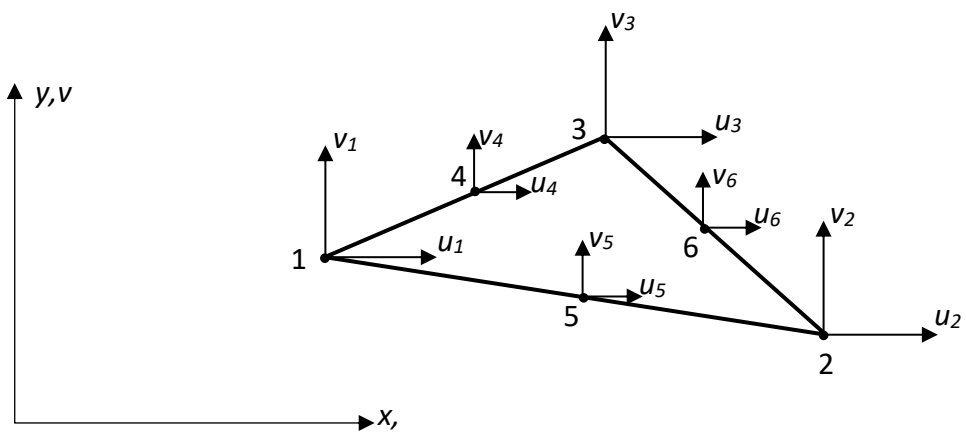


FIG 37 – NODES AND DEGREES OF FREEDOM OF THE LINEAR STRAIN TRIANGULAR ELEMENT

The nodal displacement vector is now written as:

$$\{\delta\} = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{Bmatrix} \qquad \text{Eq 243}$$

Shape Functions

Since the LST element has twelve nodal unknowns, we need twelve constants a_i to describe its complete displacement field. Usually, for the triangular elements, a complete polynomial is used in cartesian coordinates to describe the displacement field within the element. Pascal's triangle is used to assign properly the *x-y* coordinates of the polynomial to the a_i's.

1

x y

x^2 xy y^2

x^3 x^2y xy^2 y^3

x^4 x^3y x^2y^2 xy^3 y^4

x^5 x^4y x^3y^2 x^2y^3 xy^4 y^5

FIG 38 – PASCAL'S TRIANGLE FOR LST ELEMENT

So, for the LST, a quadratic displacement function is selected:

$$u(x,y) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2$$
$$v(x,y) = a_7 + a_8x + a_9y + a_{10}x^2 + a_{11}xy + a_{12}y^2 \qquad \text{Eq 244}$$

The adjacent elements exhibit a displacement compatibility guarantee by a parabola defined by three points along each side.

So, the general displacement function is written:

$$\{\Gamma\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$$
$$= \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ \cdot \\ \cdot \\ a_{12} \end{Bmatrix} \qquad \text{Eq 245}$$

In matrix form:

$$\{\Gamma\} = [\Lambda]\{a\} \qquad \text{Eq 246}$$

With

$$[\Lambda] = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{bmatrix} \qquad \text{Eq 247}$$

First, let's substitute the coordinates into *u* and *v* using the Eq 245:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_5 & y_5 & x_5^2 & x_5y_5 & y_5^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \end{Bmatrix} \qquad \text{Eq 248}$$

Or,

$$\{\delta\} = \{X\}\{a\} \qquad \text{Eq 249}$$

The a_i's are solved by:

$$\{a\} = \{X\}^{-1}\{\delta\} \qquad \text{Eq 250}$$

Then, the obtained a_i's are substituted into Eq 246 to obtain the general displacement equations in terms of the shape functions and the nodal degrees of freedom:

$$\{\Gamma\} = [\Lambda]\{X\}^{-1}\{\delta\} \qquad \text{Eq 251}$$

Then [N] is defined as:

$$[N] = [\Lambda]\{X\}^{-1} \qquad \text{Eq 252}$$

The general displacement function becomes:

$$\{\Gamma\} = [N]\{\delta\} \qquad \text{Eq 253}$$

Strain-Displacement & Stress-Strain Relationships

Knowing that the strains of the two-dimensional element are:

$$\{\epsilon\} = \begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \qquad \text{Eq 254}$$

So, using Eq 245 of *u* and *v* into Eq 254:

$$\{\varepsilon\} = \begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ . \\ . \\ a_{12} \end{Bmatrix} \qquad \text{Eq 255}$$

The equation above shows a linear strain variation in the element. In matrix form:

$$\{\varepsilon\} = [\Lambda']\{a\} \qquad \text{Eq 256}$$

Where

$$\{\Lambda'\} = \begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{bmatrix} \qquad \text{Eq 257}$$

So, substituting Eq 250 into Eq 256, we obtain $\{\varepsilon\}$ in terms of the nodal displacements:

$$\{\varepsilon\} = [\Lambda'] \{X\}^{-1}\{\delta\} \qquad \text{Eq 258}$$

Considering the matrix [B] as a function of the variables *x* and *y* and the coordinates (x_1,y_1) to (x_6,y_6):

$$\{B\} = [\Lambda'] \{X\}^{-1} \qquad \text{Eq 259}$$

Note that [B] is a 3 x 12 matrix.

From Eq 259 into Eq 258 we can write:

$$\{\varepsilon\} = [B] \{\delta\} \qquad \text{Eq 260}$$

Like it has been stated for the CST element:

$$\{\sigma\} = [D][B]\{\delta\} \qquad \text{Eq 261}$$

Where [D] is given by Eq 176 for a plane stress problem and by Eq 177 for a plane strain problem.

Element Stiffness Matrix

The stiffness matrix for the LST element can be calculated using the same equation as the one demonstrated for the CST element:

$$[k] = \iiint_V [B]^T [D][B]\, dV \qquad \text{Eq 262}$$

However, unlike the CST element, the [B] matrix of the LST element is a function of *x* and *y* but also of the nodal coordinates. The [B] matrix is defined by:

$$[B\,] = \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 & \beta_4 & 0 & \beta_5 & 0 & \beta_6 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 & 0 & \gamma_4 & 0 & \gamma_5 & 0 & \gamma_6 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 & \gamma_4 & \beta_4 & \gamma_5 & \beta_5 & \gamma_6 & \beta_6 \end{bmatrix} \qquad \text{Eq 263}$$

The β's and γ's are obtained by the inversion of the matrix [X].

So, to compute the stiffness matrix using the Eq 262, since [B] is a function of x and y, the integration needs to be done before to multiply the matrices. The resulting stiffness matrix of the LST element is therefore a 12 x 12 matrix.

It is very tedious to compute this matrix explicitly. This integration is however very easy for a computer to carry out numerically.

So, the LST stiffness matrix is of the form:

$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & . & . & . & k_{1,12} \\ k_{21} & k_{22} & k_{23} & . & . & . & k_{2,12} \\ k_{31} & k_{32} & k_{33} & . & . & . & k_{3,12} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ k_{12,1} & k_{12,2} & k_{12,3} & . & . & . & k_{12,12} \end{bmatrix} \qquad \text{Eq 264}$$

5.5.4. Thin-Plate

Assumptions and Basic Equations of the Kirchhoff Theory

The behavior of a plate is similar to that of a beam, they both carry transverse loads by bending action. To start with the plates, we will focus first on the thin plates for which the behavior is described by the Kirchhoff theory. This theory is named for the German physicist Gustav Kirchhoff. So, considering a thin-plate in the xy plane and of thickness *t* measured in the z-direction submitted to a transverse loading *q*.

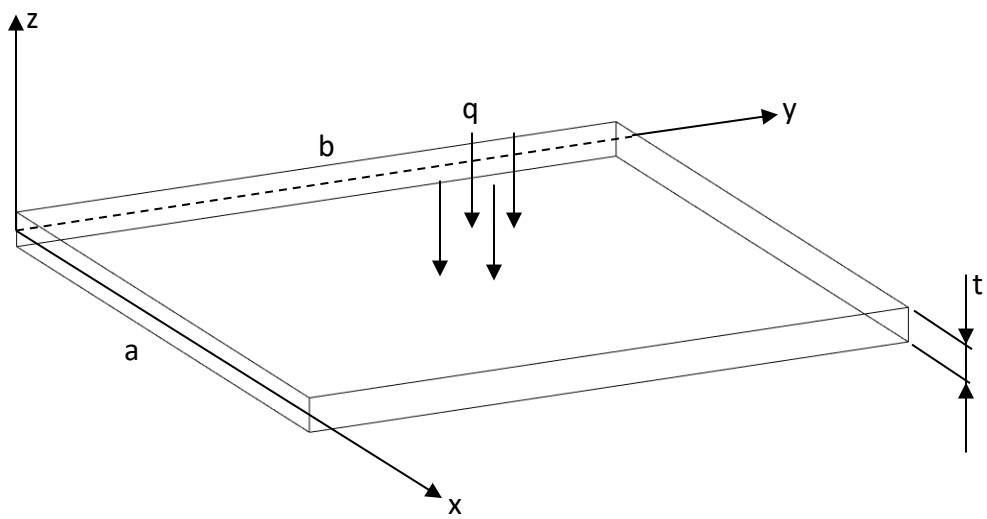


FIG 39 – THIN-PLATE GEOMETRY AND LOADING

The thin plate satisfies the following assumptions:

- The plate thickness is much smaller than the two dimensions *a* and *b*: *t << a* and *b*). For a thin plate the basic assumption is that *t* is less than 1/10 the span of the plate. If *t* is more than 1/10 the span of the plate, the transverse shear has to be accounted for and the Kirchhoff theory is no longer applicable, we are in the domain of the thick-plate theory named the Mindlin theory (see next paragraph).
- The deflection in the transverse direction, denoted *w*, is much less than the thickness *t* (*w/t<<1*).

The transverse loading *q* applied on the thin-plate produces a lateral deformation *w* in the z-direction which is a function of *x* and *y*: it is denoted *w(x,y)*.

The Kirchhoff assumptions are:

- A line D-D' drawn perpendicular to the mid-surface of the plate before deformation remains perpendicular after deformation (see Fig 40)
- The normal remain normal. That means the transverse shear strains γ_{XZ} and γ_{YZ} are null.

- The in-plane shear strain γ_{xY} is not null. In the plane <x,y>, the right angles do not remain right angles
- The variation of thickness is null: $\varepsilon_Z = 0$
- $\sigma_Z = 0$. So, the normal stress has no influence on the in-plane stress-strain equations.
- The in-plane deformations at the mid-surface are null: u(x,y,0) = v(x,y,0) = 0

Let's draw the behavior of a differential slice of plate before and after deformation following the Kirchhoff assumptions.

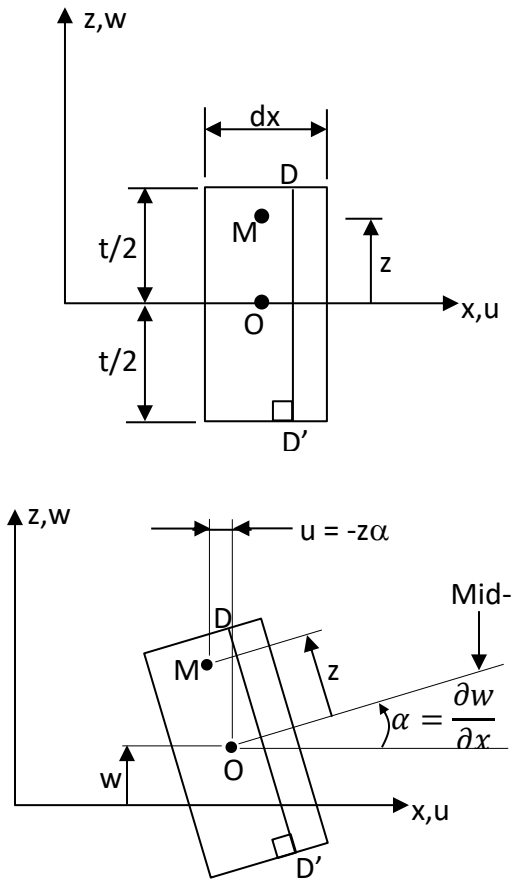


FIG 40 – BEHAVIOR OF THE DIFFERENTIAL SLICE OF PLATE ACCORDING TO THE KIRCHHOFF THEORY. [TOP] BEFORE DEFORMATION [BOTTOM] AFTER DEFORMATION

As shown in Fig 40, any point M of the plate cross-section has a *u* displacement in the x-direction due to the small rotation α:

$$u = -z\alpha = -z\frac{\partial w}{\partial x} \qquad \text{Eq 265}$$

The behavior shown in the <x,z> plane on the Fig 40 is replicated in the <y,z> plane:

$$v = -z\alpha = -z\frac{\partial w}{\partial y} \qquad \text{Eq 266}$$

So, writing the equations of the rate of change of the angular displacements of the normal known as the curvatures of the plate:

$$\chi_x = -\frac{\partial^2 w}{\partial x^2} \quad \Big| \quad \chi_y = -\frac{\partial^2 w}{\partial y^2} \quad \Big| \quad \chi_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} \qquad \text{Eq 267}$$

Since the in-plane strain-displacement equations are written:

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2} \quad \Big| \quad \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2} \quad \Big| \quad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y} \qquad \text{Eq 268}$$

By using Eq 267 into Eq 268Eq 270:

$$\varepsilon_x = -z\chi_x \quad \Big| \quad \varepsilon_y = -z\chi_y \quad \Big| \quad \gamma_{xy} = -z\chi_{xy} \qquad \text{Eq 269}$$

Based on the assumptions of the Kirchhoff theory, the in-plane stress-strain relationships for an isotropic material are:

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) \quad \Big| \quad \sigma_y = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right) \quad \Big| \quad \tau_{xy} = G\gamma_{xy} \qquad \text{Eq 270}$$

The normal stresses σ and shear stresses τ are plotted in Fig 41 acting on the edges of the plate:

- The stresses σ_X, σ_Y, τ_{XY} and τ_{YX} follow a linear variation in the z-direction from the mid-surface.
- The transverse shear stresses τ_{XZ} and τ_{YZ} are null as per the Kirchhoff assumptions.

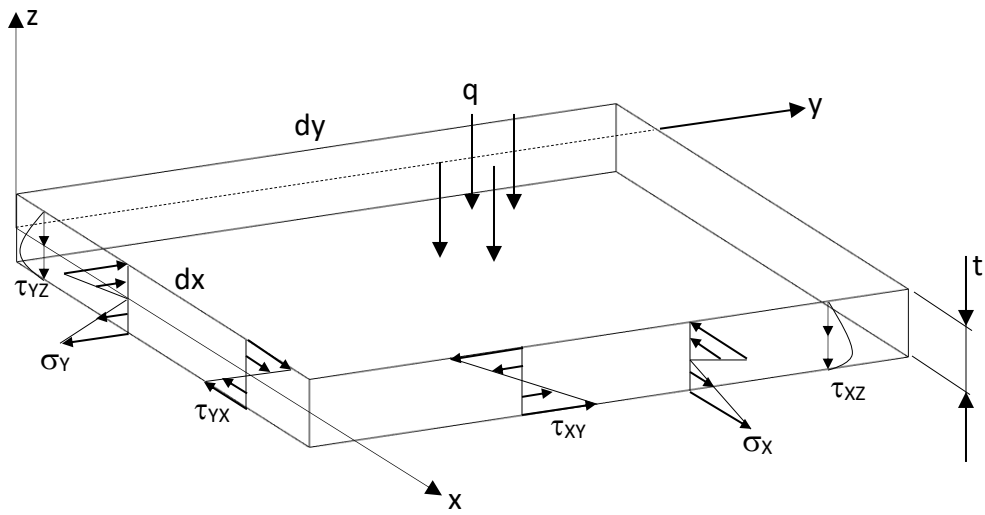


FIG 41 – STRESSES ON THE EDGES OF THE DIFFERENTIAL ELEMENT OF THE PLATE

Moreover, the bending moments M_X and M_Y as well as the twisting moment M_{XY} acting along the edges of the plate can be related to the stresses of equations Eq 270 as shown in Fig 42:

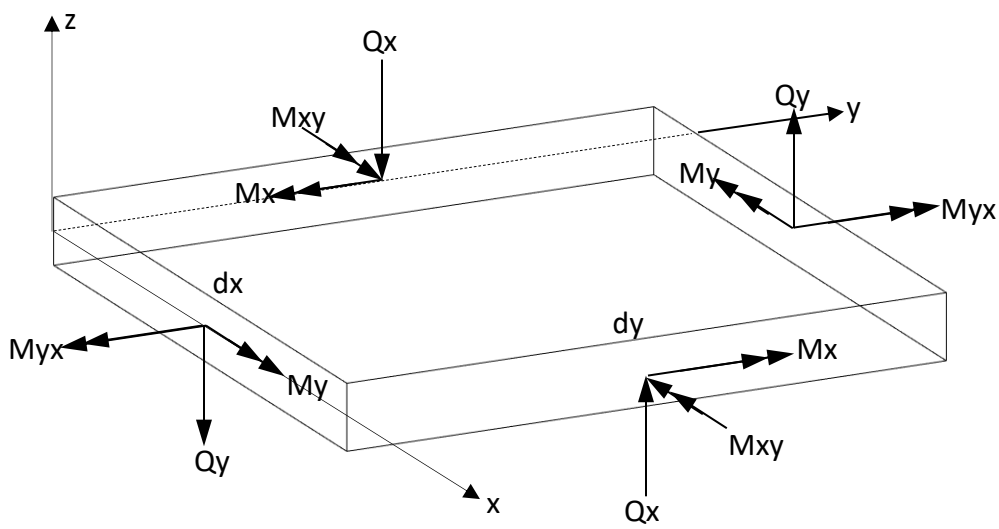


FIG 42 – FORCES AND MOMENTS ON THE EDGES OF THE DIFFERENTIAL ELEMENT OF THE PLATE

The moments per unit length functions of *x* and *y* are:

$$M_x = \int_{-t/2}^{t/2} z\sigma_x dz \quad \Bigg| \quad M_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} z\sigma_y dz \quad \Bigg| \quad M_{xy} = \int_{-t/2}^{t/2} z\tau_{xy} dz \qquad \text{Eq 271}$$

So, substituting Eq 269 & Eq 270 into Eq 271, the moments are function of the curvatures by:

$$M_x = D(\chi_x + \nu\chi_y) \quad \Big| \quad M_y = D(\chi_y + \nu\chi_x) \quad \Big| \quad M_{xy} = \frac{D(1-\nu)}{2}\chi_{xy} \qquad \text{Eq 272}$$

Where D is the bending rigidity of the plate:

$$D = \frac{Et^3}{12(1-\nu^2)} \qquad \text{Eq 273}$$

The equilibrium of forces with respect to the z-axis as well as the equilibrium of moments around the *x* and *y* axes permit to write the equilibrium equations in the form of the following differential equations:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \quad \Big| \quad \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \quad \Big| \quad \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0 \qquad \text{Eq 274}$$

So, by substituting the moment-curvature relationships Eq 272 into Eq 274 and by solving for *Qx* and *Qy*, the governing partial differential equation for an isotropic thin-plate in bending is:

$$D\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2}\right) = q \qquad \text{Eq 275}$$

This governing partial differential equation shows that the thin plate bending behavior depends only on the transverse displacement *w*.

<u>Degrees-of-freedom</u>

To derive the stiffness matrix of a thin plate we will consider the four nodes element called the quadrilateral element. Each node [j,k,m,n] of the quadrilateral element has three degrees of freedom:

- A transverse displacement *w* the z-direction
- A rotation θx about the x-axis
- A rotation θy about the y-axis

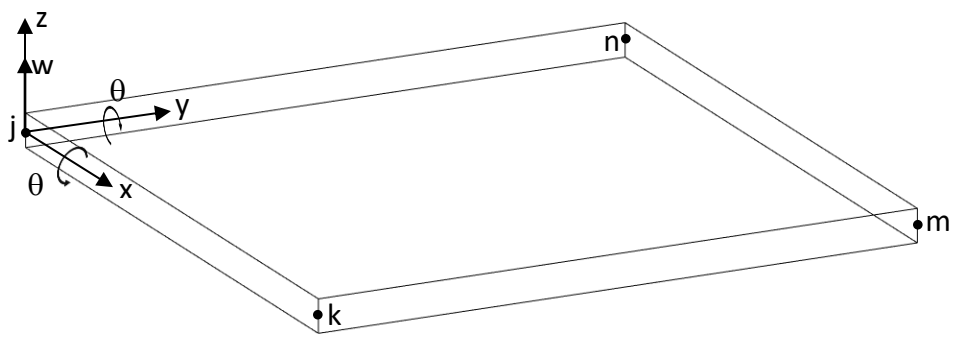


FIG 43 – THIN-PLATE DEGREES OF FREEDOM

The nodal displacement vector at node i (for i = j, k, m or n) is:

$$\{\delta_i\} = \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} \qquad \text{Eq 276}$$

Note that the rotations θ are related to the transverse displacement by:

$$\theta_x = \frac{\partial w}{\partial y}$$
$$\theta_y = -\frac{\partial w}{\partial x} \qquad \text{Eq 277}$$

The negative sign for θy is needed because to produce a positive rotation about the y-axis, a negative displacement *w* is required.

The total quadrilateral element displacement vector is composed with 4 x 3 = 12 degrees of freedom:

$$\{\delta\} = \begin{Bmatrix} w_j \\ \theta_{xj} \\ \theta_{yj} \\ w_k \\ \theta_{xk} \\ \theta_{yk} \\ w_m \\ \theta_{xm} \\ \theta_{ym} \\ w_n \\ \theta_{xn} \\ \theta_{yn} \end{Bmatrix} = \begin{Bmatrix} \delta_j \\ \delta_k \\ \delta_m \\ \delta_n \end{Bmatrix} \qquad \text{Eq 278}$$

Shape Functions

Since the thin-plate element has twelve degrees of freedom, we need twelve constants a_i to describe its complete displacement field with a polynomial function. Like for the LST elements, a complete polynomial in cartesian coordinates is used to describe the displacement field within the element. Pascal's triangle is used to assign properly the *x-y* coordinates of the polynomial to the a_i's.

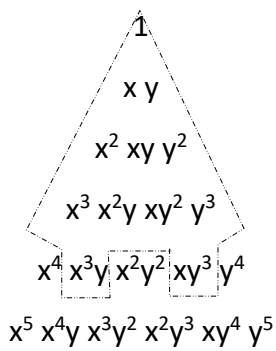


FIG 44 – PASCAL'S TRIANGLE FOR THE THIN-PLATE ELEMENT

The last two terms x^3y and xy^3 are chosen to ensure the continuity in displacement at the boundaries of element. The x^2y^2 is rejected because it cannot be paired with any other terms.

$$w = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + a_8x^2y + a_9xy^2 + a_{10}y^3 + a_{11}x^3y + a_{12}xy^3 \qquad \text{Eq 279}$$

Similarly to the 1D beam element, the constants a_i's can be computed by expressing the 12 simultaneous shape functions linking the values of *w* and its slopes at the four nodes of the quadrilateral element using their appropriate values.

Starting with the evaluation of the slopes:

$$\frac{\partial w}{\partial y} = a_3 + a_5x + 2a_6y + a_8x^2 + 2a_9xy + 3a_{10}y^2 + a_{11}x^3 + 3a_{12}xy^2$$

$$\frac{\partial w}{\partial x} = a_2 + 2a_4x + a_5y + 3a_7x^2 + 2a_8xy + a_9y^2 + 3a_{11}x^2y + a_{12}y^3 \qquad \text{Eq 280}$$

So, Eq 279 & Eq 280 in matrix form:

$$\begin{Bmatrix} w \\ \dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial x} \end{Bmatrix} = \begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} =$$

$$= \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3 \\ 0 & 0 & 1 & 0 & x & 2y & 0 & x^2 & 2xy & 3y^2 & x^3 & 3xy^2 \\ 0 & -1 & 0 & -2x & -y & 0 & -3x^2 & -2xy & -y^2 & 0 & -3x^2y & -y^3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ . \\ . \\ a_{12} \end{Bmatrix} \quad \text{Eq 281}$$

Again, the general displacement function Γ is expressed as:

$$\{\Gamma\} = [\Lambda]\{a\} \quad \text{Eq 282}$$

With

$$[\Lambda] = \\ = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3 \\ 0 & 0 & 1 & 0 & x & 2y & 0 & x^2 & 2xy & 3y^2 & x^3 & 3xy^2 \\ 0 & -1 & 0 & -2x & -y & 0 & -3x^2 & -2xy & -y^2 & 0 & -3x^2y & -y^3 \end{bmatrix} \quad \text{Eq 283}$$

Eq 281 is now expanded by using the appropriate values at each point of the quadrilateral element:

$$\{\delta\} = \begin{Bmatrix} w_j \\ \theta_{xj} \\ \theta_{yj} \\ w_k \\ \theta_{xk} \\ \theta_{yk} \\ w_m \\ \theta_{xm} \\ \theta_{ym} \\ w_n \\ \theta_{xn} \\ \theta_{yn} \end{Bmatrix} =$$

$$= \begin{bmatrix} 1 & x_j & y_j & x_j^2 & x_j y_j & {y_j}^2 & {x_j}^3 & {x_j}^2 y_j & x_j {y_j}^2 & {y_j}^3 & {x_j}^3 y_j & x_j {y_j}^3 \\ 0 & 0 & 1 & 0 & x_j & 2y_j & 0 & {x_j}^2 & 2x_j y_j & 3{y_j}^2 & {x_j}^3 & 3x_j {y_j}^2 \\ 0 & -1 & 0 & -2x_j & -y_j & 0 & -3{x_j}^2 & -2x_j y_j & -{y_j}^2 & 0 & -3{x_j}^2 y_j & -{y_j}^3 \\ . & & & & & & & & & & & . \\ . & & & & & & & & & & & . \\ . & & & & & & & & & & & . \\ . & & & & & & & & & & & . \\ . & & & & & & & & & & & . \\ . & & & & & & & & & & & . \\ 1 & x_n & y_n & x_n^2 & x_n y_n & {y_n}^2 & {x_n}^3 & {x_n}^2 y_n & x_n {y_n}^2 & {y_n}^3 & {x_n}^3 y_n & x_n {y_n}^3 \\ 0 & 0 & 1 & 0 & x_n & 2y_n & 0 & {x_n}^2 & 2x_n y_n & 3{y_n}^2 & {x_n}^3 & 3x_n {y_n}^2 \\ 0 & -1 & 0 & -2x_n & -y_n & 0 & -3{x_n}^2 & -2x_n y_n & -{y_n}^2 & 0 & -3{x_n}^2 y_n & -{y_n}^3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \end{Bmatrix}$$

Eq 284

$$\{\delta\} = [\mathrm{X}]\{a\}$$

Eq 285

The twelve constants a_i are solved by:

$$\{a\} = [X]^{-1}\{\delta\}$$

Eq 286

Substituting Eq 286 into Eq 282:

$$\{\Gamma\} = [\Lambda][X]^{-1}\{\delta\}$$

Eq 287

With the shape function matrix given by:

$$\{\mathrm{N}\} = [\Lambda][X]^{-1}$$

Eq 288

the general displacement function Γ is written in function of the shape function matrix [N] and the nodal displacement vector {δ}:

$$\{\Gamma\} = [\mathrm{N}]\{\delta\}$$

Eq 289

Curvature-Displacement & Moment-Curvature Relationships

As per the Kirchhoff assumptions we have:

$$\chi_x = -\frac{\partial^2 w}{\partial x^2} \quad \Bigg| \quad \chi_y = -\frac{\partial^2 w}{\partial y^2} \quad \Bigg| \quad \chi_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} \qquad \text{Eq 290}$$

So, by evaluating the above second derivatives of *w* from Eq 279, the curvature matrix is:

$$\begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & -6x & -2y & 0 & 0 & -6xy & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -2x & -6y & 0 & -6xy \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & -4x & -4y & 0 & -6x^2 & -6y^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \end{Bmatrix} \qquad \text{Eq 291}$$

$$\{\chi\} = [\Lambda']\{a\} \qquad \text{Eq 292}$$

Substituting Eq 286 into Eq 292:

$$\{\chi\} = [\Lambda'][X]^{-1}\{\delta\} \qquad \text{Eq 293}$$

So, by considering the gradient matrix [B]:

$$[B] = [\Lambda'][X]^{-1} \qquad \text{Eq 294}$$

The curvature matrix is given by:

$$\{\chi\} = [B]\{\delta\} \qquad \text{Eq 295}$$

By writing Eq 272 & Eq 273 in matrix form:

$$\{M\} = \begin{Bmatrix} Mx \\ My \\ Mxy \end{Bmatrix} = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} \qquad \text{Eq 296}$$

Or,

$$\{M\} = [D]\{\chi\} \qquad \text{Eq 297}$$

Where [D] is the constitutive matrix for an isotropic material:

$$[D] = \frac{Et^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad \text{Eq 298}$$

Finally, using Eq 295 into Eq 297:

$$\{M\} = [D][B]\{\delta\} \qquad \text{Eq 299}$$

<u>Element Stiffness Matrix</u>

The stiffness matrix of the four nodes thin-plate element is calculated using the usual form:

$$[k] = \iint [B]^T[D][B]dxdy \qquad \text{Eq 300}$$

Where:

- [B] is computed by Eq 294
- [D] is computed by Eq 298

The element stiffness matrix [k] is a 12 x 12 matrix.

$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & . & . & . & k_{1,12} \\ k_{21} & k_{22} & k_{23} & . & . & . & k_{2,12} \\ k_{31} & k_{32} & k_{33} & . & . & . & k_{3,12} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ k_{12,1} & k_{12,2} & k_{12,3} & . & . & . & k_{12,12} \end{bmatrix} \qquad \text{Eq 301}$$

5.5.5. Isoparametric Formulation

The development of stiffness matrices for the membranes CST, LST as well as the four nodes plate elements presented in the previous paragraphs have been done using a global coordinate system. We have seen that the developments of the equations in terms of global coordinates is tedious. However, an isoparametric formulation is available which leads to a simpler formulation more adequate for programming and computing. This formulation is generally used for 2D and 3D stress analysis. Moreover, the isoparametric formulation offers the possibility to have non-right-angles elements or elements with curved sides. This formulation has been implemented in all the modern FEA packages. This formulation is used for the derivation of the thick-plate element stiffness matrix using the Mindlin assumptions. But before, the development of the stiffness matrix for the membrane four nodes quadrilateral element using the isoparametric formulation is required.

<u>The Natural Coordinate System</u>

Since the quadrilateral element may be non-rectangular, an alternative coordinate system called the natural coordinate system <ξ, η> is introduced:

- Its origin in the global coordinates <x,y> is at the average of the corner coordinates
- Regardless of the shape of the element, the physical size of the element or its orientation in the global coordinate system <x,y>, the edges and the corners are at ξ = ±1 and η = ±1
- ξ and η are not necessarily orthogonal and have no particular orientation with respect to axes x and y

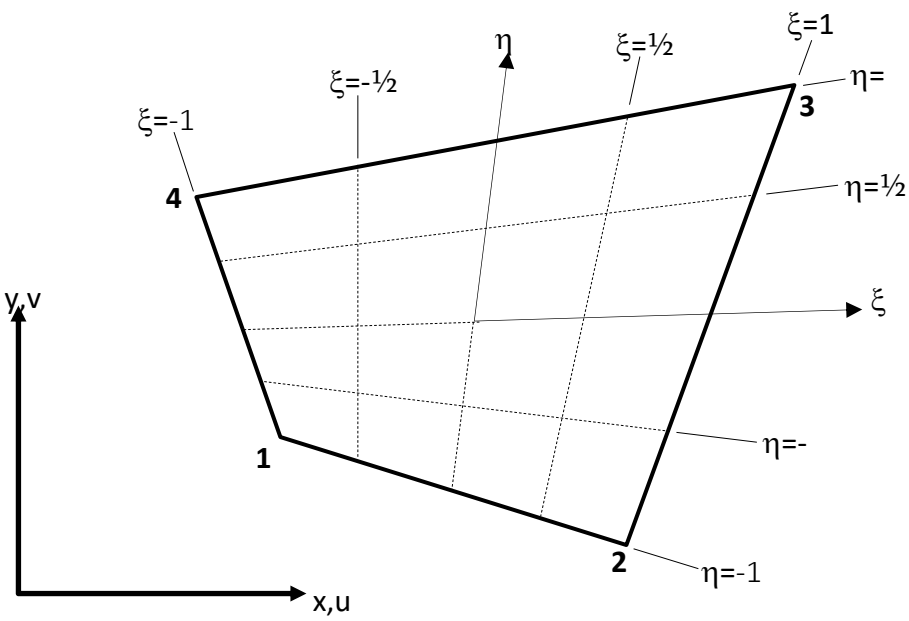


FIG 45 – FOUR NODES QUADRILATERAL ELEMENT IN THE <ξ, η> NATURAL COORDINATE SYSTEM

Degrees of Freedom

Let's consider the membrane four nodes quadrilateral element having two degrees of freedom (u_i, v_i) at each node in the global coordinate system <x,y>. So, the membrane quadrilateral element has eight degrees of freedom.

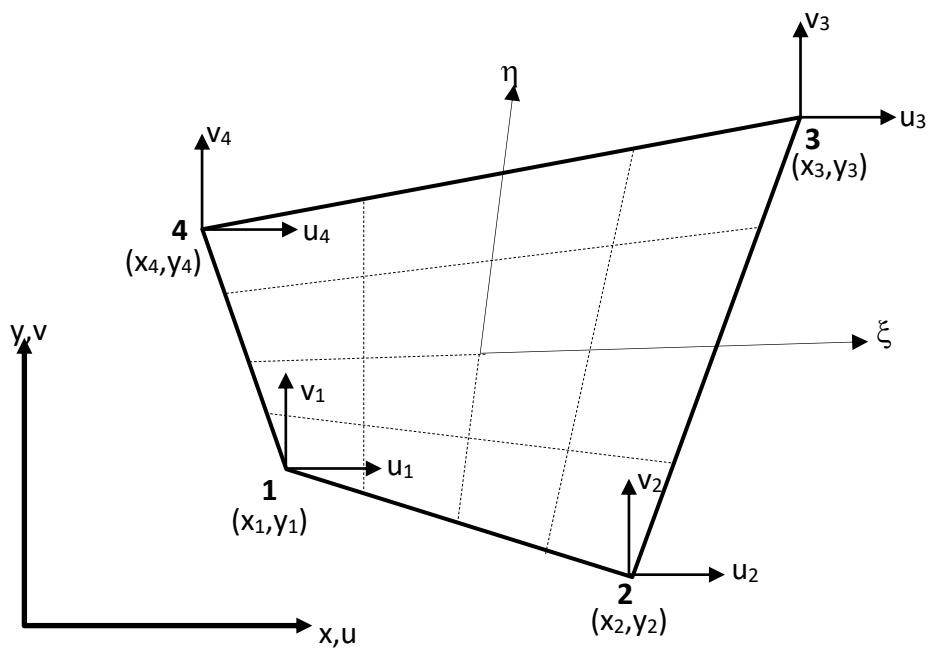


FIG 46 – DEGREES OF FREEDOM OF THE MEMBRANE FOUR NODES QUADRILATERAL ELEMENT

Shape Functions

The shape functions will be defined to map the quadrilateral element of Fig 45 in the natural coordinates <ξ, η>. Since the membrane quadrilateral element has eight degrees of freedom, we need eight constants a_i to describe its complete displacement field.

First, the element shape described in the x and y coordinates can be related to the isoparametric coordinates ξ and η by:

$$x = a_1 + a_2\xi + a_3\eta + a_4\xi\eta$$
$$y = a_5 + a_6\xi + a_7\eta + a_8\xi\eta \qquad \text{Eq 302}$$

As done previously, the a_i's coefficients in terms of the eight known nodal coordinates (x_1,y_1), (x_2,y_2), (x_3,y_3) & (x_4,y_4) can be solved:

$$x = \frac{1}{4}[(1-\xi)(1-\eta)x_1 + (1+\xi)(1-\eta)x_2 + (1+\xi)(1+\eta)x_3 + (1-\xi)(1+\eta)x_4] \qquad \text{Eq 303}$$

$$y = \frac{1}{4}[(1-\xi)(1-\eta)y_1 + (1+\xi)(1-\eta)y_2 + (1+\xi)(1+\eta)y_3 + (1-\xi)(1+\eta)y_4]$$

In matrix form:

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix} \qquad \text{Eq 304}$$

With the shape functions of the above equation:

$$N_1 = \frac{(1-\xi)(1-\eta)}{4}$$
$$N_2 = \frac{(1+\xi)(1-\eta)}{4}$$
$$N_3 = \frac{(1+\xi)(1+\eta)}{4}$$
$$N_4 = \frac{(1-\xi)(1+\eta)}{4} \qquad \text{Eq 305}$$

So, the displacement function Γ can now be defined by the same shape functions used to define the element shape:

$$\{\Gamma\} = [\mathrm{N}]\{\delta\} \qquad \text{Eq 306}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \qquad \text{Eq 307}$$

<u>Strain-Displacement & Stress-Strain Relationships</u>

To calculate the stiffness matrix [K], the strain-displacement matrix [B] has to be computed. It is easier to formulate [B] in terms of natural coordinates ξ and η rather than in the cartesian

coordinates. Moreover, this isoparametric approach is easier to program and it significantly improves computing time.

The strains are defined in terms of the derivatives of the displacements with respect to *x* and *y* coordinates (for example $\varepsilon_x=\partial u/\partial x$). However, the displacements *u* and *v* are now functions of ξ and η coordinates as given by Eq 307 with the shape functions given by Eq 305.

So, to generalize the problem, let introduce a function *j* (could be *u* or *v*) expressed in terms of ξ and η for which the derivatives are computed with respect to *x* and *y*. So, the chain rule of differentiation is applied since it is not possible to express ξ and η as functions of *x* and *y* directly. For *j* as a function of *x* and *y*, the chain rule gives:

$$\frac{\partial j}{\partial \xi}=\frac{\partial j}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial j}{\partial y}\frac{\partial y}{\partial \xi}$$
$$\frac{\partial j}{\partial \eta}=\frac{\partial j}{\partial x}\frac{\partial x}{\partial \eta}+\frac{\partial j}{\partial y}\frac{\partial y}{\partial \eta} \qquad \text{Eq 308}$$

Since the function *j* can be the displacement functions *u* or *v*, $(\partial j/\partial \xi)$ and $(\partial j/\partial \eta)$ are computed by Eq 307 while $(\partial x/\partial \xi)$, $(\partial x/\partial \eta)$, $(\partial y/\partial \xi)$ and $(\partial y/\partial \eta)$ are computed using Eq 304. The two-remaining unknown in the Eq 308 are $(\partial j/\partial x)$ and $(\partial j/\partial x)$ needed to compute the strains. To solve the Eq 308 for $(\partial j/\partial x)$ and $(\partial j/\partial x)$ the Cramer's rule is proposed.

<u>Reminder on the Cramer's rule:</u>

To solve the following system for {X}:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}=\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} \qquad \text{Eq 309}$$

Where:

- [K] is a 3 x 3 known matrix.
- {f} is a known vector.
- {X} is the unknown vector.

To solve the system Eq 309 for {X}, the following Cramer's rule is applied in two steps:

- Step 1: Create the matrix $[L_i]$ as the matrix [K] with column *i* replaced by the vector {f}
- Step 2: Compute:

$$x_i=\frac{|L_i|}{|K|} \qquad \text{Eq 310}$$

Where,

- $|L_i|$ is the determinant of matrix $[L_i]$
- $|K|$ is the determinant of matrix $[K]$

So, by applying the Cramer's rule, the system Eq 309 for {X} is solved:

$$x_1 = \frac{|L_1|}{|K|} = \frac{\begin{vmatrix} f_1 & K_{12} & K_{13} \\ f_2 & K_{22} & K_{23} \\ f_3 & K_{32} & K_{33} \end{vmatrix}}{\begin{vmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{vmatrix}} \qquad x_2 = \frac{|L_2|}{|K|} = \frac{\begin{vmatrix} K_{11} & f_1 & K_{13} \\ K_{21} & f_2 & K_{23} \\ K_{31} & f_3 & K_{33} \end{vmatrix}}{\begin{vmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{vmatrix}}$$

$$x_3 = \frac{|L_3|}{|K|} = \frac{\begin{vmatrix} K_{11} & K_{12} & f_1 \\ K_{21} & K_{22} & f_2 \\ K_{31} & K_{32} & f_3 \end{vmatrix}}{\begin{vmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{vmatrix}} \qquad \text{Eq 311}$$

Using the Cramer's rule, Eq 308 is solved for ($\partial j/\partial x$) and ($\partial j/\partial x$):

$$\frac{\partial j}{\partial x} = \frac{\begin{vmatrix} \frac{\partial j}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial j}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix}} \qquad \frac{\partial j}{\partial y} = \frac{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial j}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial j}{\partial \eta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix}} \qquad \text{Eq 312}$$

Where the denominator matrix is called the Jacobian matrix [J]. So, the Jacobian matrix is given by:

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \qquad \text{Eq 313}$$

Let's continue with the expression of the element strains already used in previous paragraphs:

$$\epsilon = [B]\{\delta\} \qquad \text{Eq 314}$$

Here, [B] is expressed as a function of ξ and η. The usual strain-displacement relationships are:

$$\{\epsilon\} = \begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \qquad \text{Eq 315}$$

Let's introduce an operator matrix to re-write the Eq 315:

$$\begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial(\quad)}{\partial x} & 0 \\ 0 & \dfrac{\partial(\quad)}{\partial y} \\ \dfrac{\partial(\quad)}{\partial y} & \dfrac{\partial(\quad)}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad \text{Eq 316}$$

Where the operator matrix is

$$[\mathrm{E}] = \begin{bmatrix} \dfrac{\partial(\quad)}{\partial x} & 0 \\ 0 & \dfrac{\partial(\quad)}{\partial y} \\ \dfrac{\partial(\quad)}{\partial y} & \dfrac{\partial(\quad)}{\partial x} \end{bmatrix} \qquad \text{Eq 317}$$

In the operator matrix epsilon from Eq 317, ∂()/∂x and ∂()/∂y are the partial derivatives of any variable we included in the parentheses.

Coming back to Eq 312 the determinants in the numerators are evaluated:

$$\frac{\partial(\quad)}{\partial x} = \frac{1}{|J|}\left[\frac{\partial y}{\partial \eta}\frac{\partial(\quad)}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial(\quad)}{\partial \eta}\right]$$
$$\frac{\partial(\quad)}{\partial y} = \frac{1}{|J|}\left[\frac{\partial x}{\partial \xi}\frac{\partial(\quad)}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial(\quad)}{\partial \xi}\right] \qquad \text{Eq 318}$$

In the Eq 318 above, |J| is the determinant of the Jacobian matrix given by Eq 313.

So, by substituting Eq 318 into Eq 316, the strains are expressed in terms of the natural coordinates ξ and η:

$$\begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} \left[\frac{\partial y}{\partial \eta}\frac{\partial(\quad)}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial(\quad)}{\partial \eta}\right] & 0 \\ 0 & \left[\frac{\partial x}{\partial \xi}\frac{\partial(\quad)}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial(\quad)}{\partial \xi}\right] \\ \left[\frac{\partial x}{\partial \xi}\frac{\partial(\quad)}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial(\quad)}{\partial \xi}\right] & \left[\frac{\partial y}{\partial \eta}\frac{\partial(\quad)}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial(\quad)}{\partial \eta}\right] \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad \text{Eq 319}$$

By using Eq 307 the strains of Eq 319 are expressed in terms of the shape functions:

$$\begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} \left[\frac{\partial y}{\partial \eta}\frac{\partial(\quad)}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial(\quad)}{\partial \eta}\right] & 0 \\ 0 & \left[\frac{\partial x}{\partial \xi}\frac{\partial(\quad)}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial(\quad)}{\partial \xi}\right] \\ \left[\frac{\partial x}{\partial \xi}\frac{\partial(\quad)}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial(\quad)}{\partial \xi}\right] & \left[\frac{\partial y}{\partial \eta}\frac{\partial(\quad)}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial(\quad)}{\partial \eta}\right] \end{bmatrix}$$

$$* \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \qquad \text{Eq 320}$$

Or in compact matrix form:

$$\{\varepsilon\} = [\Psi][N]\{\delta\} \qquad \text{Eq 321}$$

Where $[\Psi]$ is another operator matrix given by:

$$[\Psi] = \frac{1}{|J|} \begin{bmatrix} \left[\frac{\partial y}{\partial \eta}\frac{\partial(\quad)}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial(\quad)}{\partial \eta}\right] & 0 \\ 0 & \left[\frac{\partial x}{\partial \xi}\frac{\partial(\quad)}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial(\quad)}{\partial \xi}\right] \\ \left[\frac{\partial x}{\partial \xi}\frac{\partial(\quad)}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial(\quad)}{\partial \xi}\right] & \left[\frac{\partial y}{\partial \eta}\frac{\partial(\quad)}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial(\quad)}{\partial \eta}\right] \end{bmatrix} \qquad \text{Eq 322}$$

[N] is the 2 x 8 shape function matrix:

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \qquad \text{Eq 323}$$

{δ} is the degrees of freedom vector:

$$\{\delta\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \qquad \text{Eq 324}$$

So, as per Eq 314, [B] is the 3 x 8 matrix defined as:

$$[B] = [\Psi][N] \qquad \text{Eq 325}$$

By substituting Eq 322 for [Ψ] into Eq 325 and Eq 305 & Eq 323 for the shape functions into Eq 325, [B] is:

$$[B(\xi,\eta)] = \frac{1}{|J|}[B_1 \quad B_2 \quad B_3 \quad B_4] \qquad \text{Eq 326}$$

Where the submatrices [B_i] are:

For i = 1, 2, 3, 4

$$[B_i] = \begin{bmatrix} \alpha\frac{\partial N_i}{\partial \xi} - \beta\frac{\partial N_i}{\partial \eta} & 0 \\ 0 & \gamma\frac{\partial N_i}{\partial \eta} - \lambda\frac{\partial N_i}{\partial \xi} \\ \gamma\frac{\partial N_i}{\partial \eta} - \lambda\frac{\partial N_i}{\partial \xi} & \alpha\frac{\partial N_i}{\partial \xi} - \beta\frac{\partial N_i}{\partial \eta} \end{bmatrix} \qquad \text{Eq 327}$$

$$\begin{aligned} \alpha &= \frac{1}{4}[(\xi-1)y_1 - (\xi+1)y_2 + (\xi+1)y_3 + (1-\xi)y_4] \\ \beta &= \frac{1}{4}[(\eta-1)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4] \\ \gamma &= \frac{1}{4}[(\eta-1)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4] \\ \lambda &= \frac{1}{4}[(\xi-1)x_1 - (\xi+1)x_2 + (\xi+1)x_3 + (1-\xi)x_4] \end{aligned} \qquad \text{Eq 328}$$

So, to evaluate [B], the determinant of the Jacobian matrix which is a polynomial of ξ and η is evaluated. Then, from Eq 313 for [J] and Eq 303 for *x* and *y*, the determinant of the Jacobian matrix is calculated:

$$|J| = \frac{1}{8}[x_1 \quad x_2 \quad x_3 \quad x_4]\begin{bmatrix} 0 & 1-\eta & \eta-\xi & \xi-1 \\ \eta-1 & 0 & \xi+1 & -\xi-\eta \\ \xi-\eta & -\xi-1 & 0 & \eta+1 \\ 1-\xi & \xi+\eta & -\eta-1 & 0 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} \qquad \text{Eq 329}$$

So, |J| and [B] are function of ξ and η and of the known global coordinates x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4. Due to the stress-strain relationship, the stress matrix σ is also a function of ξ and η:

$$\sigma = [D][B]\{\delta\} \qquad \text{Eq 330}$$

<u>Element Stiffness Matrix</u>

The stiffness matrix [k] is evaluated using the well-known equation:

$$[k] = \iiint_V [B]^T[D][B]\, dV \qquad \text{Eq 331}$$

So, for a planar element having a constant thickness *t*, Eq 331 becomes:

$$[k] = \iint_A [B]^T[D][B].t.dxdy \qquad \text{Eq 332}$$

To express the above equation in terms of <ξ,η> coordinates, since [B] is a function of ξ and η, the integration is done with respect to ξ and η. So, using the determinant of the Jacobian matrix, the transformation of *x* and *y* variables into the <ξ,η> coordinates is performed. From a theorem of integral calculus, the generic form for a function *j* is written as (see a mathematics book for the complete proof):

$$\iint_A j(x,y)dxdy = \iint_A j(\xi,\eta)|J|d\xi d\eta \qquad \text{Eq 333}$$

So, using Eq 333 into Eq 347, the iso-parametric coordinates is:

$$[k] = \int_{-1}^{+1}\int_{-1}^{+1} [B]^T[D][B]t|J|d\xi d\eta \qquad \text{Eq 334}$$

The above integration to determine the stiffness matrix is done numerically using the Gaussian quadrature method. Refer to a mathematics book for details about this method. Note that the stiffness matrix computed with Eq 334 is an 8 x 8 matrix.

5.5.6. Thick-Plate

Assumptions and Basic Equations of the Mindlin Theory

This theory is named for the American mechanical engineer Raymond David Mindlin. The Mindlin theory of plates is an extension of the Kirchhoff plate theory that considers shear deformations through-the thickness of a plate. The Mindlin theory is to the plates what the theory of Timoshenko is to the beams.

	1D Beam	**2D Plate**	**Cross-Sections**
Shear deformation neglected	EULER-BERNOULLI	KIRCHHOFF	Right angles preserved
Shear deformation taken into account	TIMOSHENKO	MINDLIN	Right angles NOT preserved

Similarly to the Timoshenko theory for the beams, the transverse shear deformation effects are included in the Mindlin plate theory. It means that the plane sections do not remain perpendicular to the mid-surface after deformation. In other words, the planes initially normal to the mid-surface may experience different rotations than the mid-surface itself. The main consequence is that the transverse strains γ_{xz} and γ_{yz} are no more null and the rotations of the cross-sections are treated as independent variables.

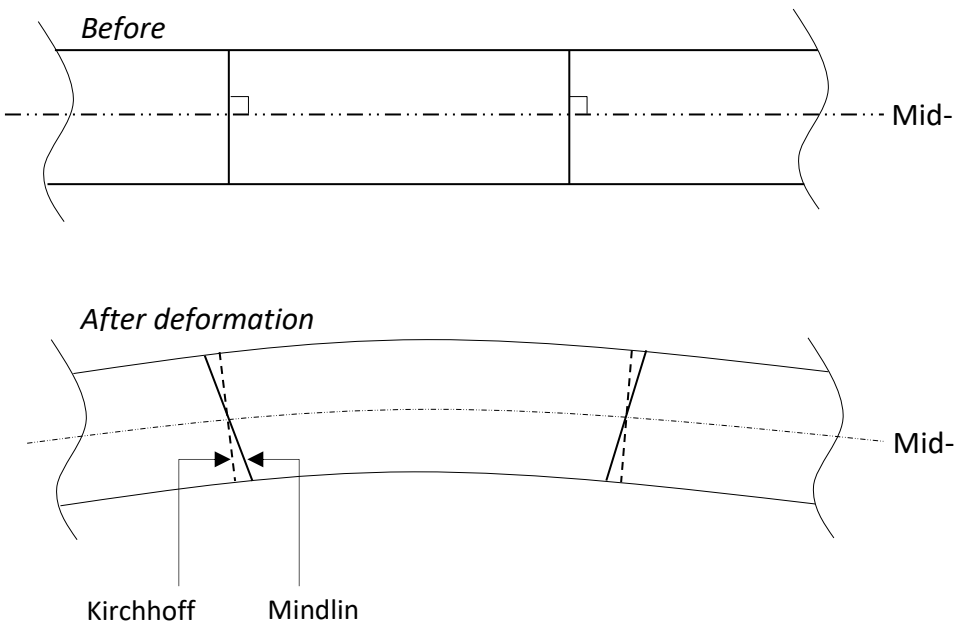


FIG 47 – SHEAR DEFORMATION IN A MINDLIN PLATE

Let's draw the behavior of a differential slice of plate after deformation following the Mindlin assumptions. Fig 48 shows the displacements of a differential slice of plate in the <x,z> and <y,z> planes. θx and θy are the rotations of lines initially perpendicular to the middle surface. The top view shows the definition of positive rotations viewed normal to <x,y> plane. The middle view shows the displacements in <x,z> plane. The bottom view shows the displacements in <y,z> plane.

The Mindlin assumptions are:

- The plate thickness is approximately one tenth of the planar dimensions
- The normal to the mid-surface remains straight but not necessarily perpendicular to the mid-surface
- The displacement field across the plate thickness is linear
- The plate thickness does not change during deformation
- The normal stress σ_Z through the thickness is ignored (plane stress condition)

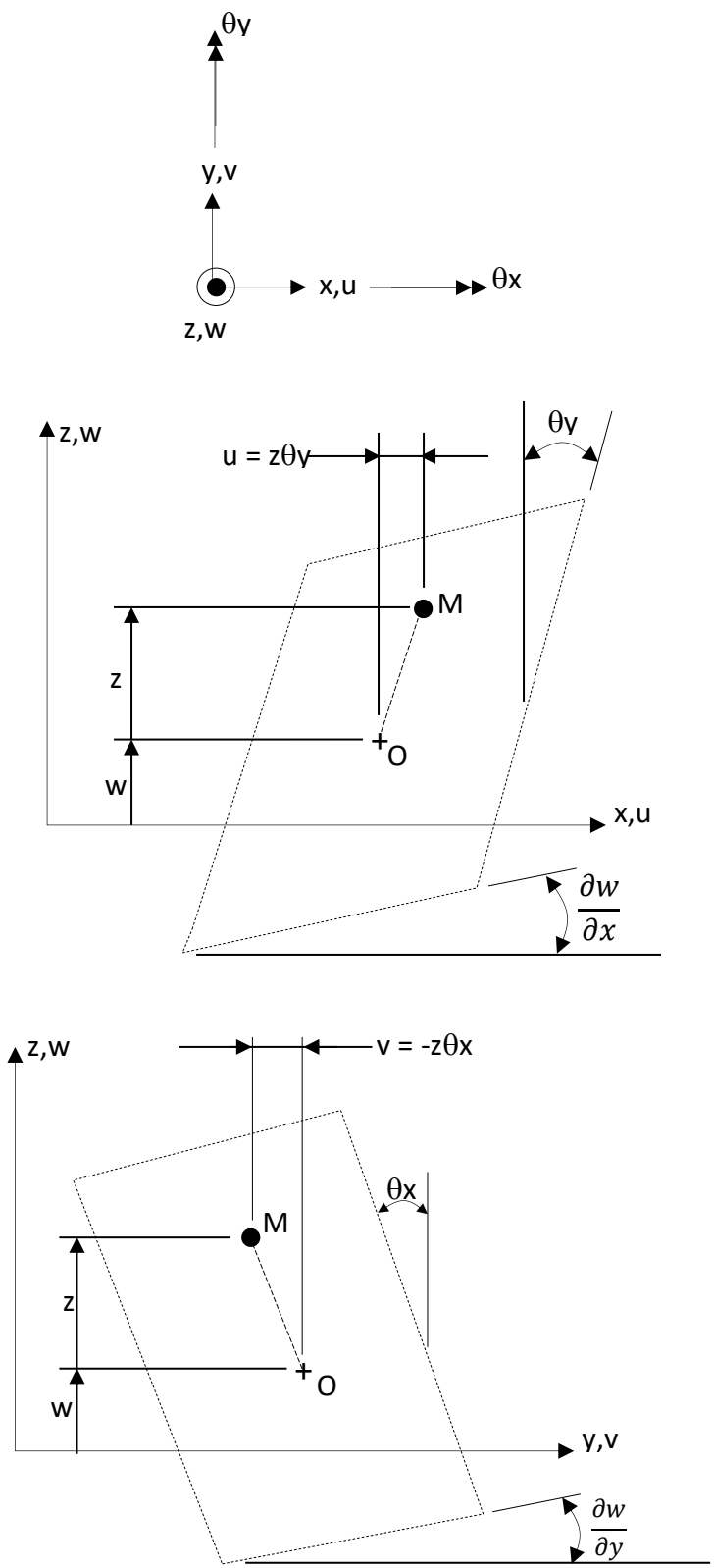


FIG 48 – BEHAVIOR OF THE DIFFERENTIAL SLICE OF PLATE ACCORDING TO THE MINDLIN THEORY

Degrees-of-freedom

To derive the stiffness matrix of the thick-plate, the same four nodes element as in thin-plate is considered. Each node [j,k,m,n] of the quadrilateral element has three degrees of freedom:

- A transverse displacement *w* in the z-direction
- A rotation θx about the x-axis
- A rotation θy about the y-axis

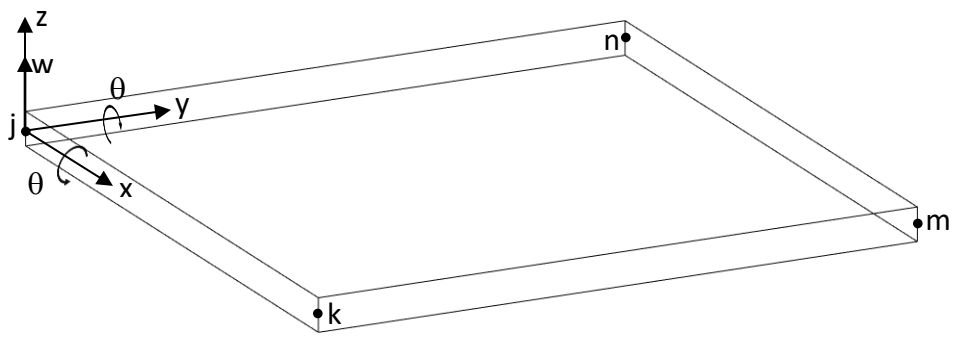


FIG 49 – THICK-PLATE DEGREES OF FREEDOM

The nodal displacement vector at node i (for i = j, k, m or n) is:

$$\{\delta_i\} = \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} \qquad \text{Eq 335}$$

The total quadrilateral element displacement vector is composed with 4 x 3 = 12 degrees of freedom:

$$\{\delta\} = \begin{Bmatrix} w_j \\ \theta_{xj} \\ \theta_{yj} \\ w_k \\ \theta_{xk} \\ \theta_{yk} \\ w_m \\ \theta_{xm} \\ \theta_{ym} \\ w_n \\ \theta_{xn} \\ \theta_{yn} \end{Bmatrix} = \begin{Bmatrix} \delta_j \\ \delta_k \\ \delta_m \\ \delta_n \end{Bmatrix} \qquad \text{Eq 336}$$

Shape Functions

Using the isoparametric formulation of the shape functions defined in the previous paragraph:

$$N_1 = \frac{(1-\xi)(1-\eta)}{4}$$
$$N_2 = \frac{(1+\xi)(1-\eta)}{4}$$
$$N_3 = \frac{(1+\xi)(1+\eta)}{4}$$
$$N_4 = \frac{(1-\xi)(1+\eta)}{4}$$

Eq 337

And, considering the four nodes isoparametric element, the variations of displacement *w* and rotations θx and θy within the thick-plate element are expressed in relation of nodal values as:

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \\ w_3 \\ \theta_{x3} \\ \theta_{y3} \\ w_4 \\ \theta_{x4} \\ \theta_{y4} \end{Bmatrix}$$

Eq 338

where,

$$w = \sum_{i=1}^{4} N_i w_i \quad \Bigg| \quad \theta_x = \sum_{i=1}^{4} N_i \theta_{xi} \quad \Bigg| \quad \theta_y = \sum_{i=1}^{4} N_i \theta_{yi}$$

Eq 339

Curvature-Displacement, Shear-Strain & Moment-Curvature Relationships

Thus, according to the Mindlin plate theory, the deformations *u* and *v* parallel to the undeformed mid-surface, at a distance *z* from the centroidal axis are given by:

$$u = z\theta_y \quad \Bigg| \quad v = -z\theta_x$$

Eq 340

Where θx and θy are the rotations of the normal line initially perpendicular to the mid-surface of the plate with respect to the *x* and *y* axes.

The curvatures are given by:

$$\chi_x = \frac{\partial \theta_y}{\partial x} \quad \Bigg| \quad \chi_y = -\frac{\partial \theta_x}{\partial y} \quad \Bigg| \quad \chi_{xy} = \frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x} \qquad \text{Eq 341}$$

Similarly to the thin-plate, the moments and the curvatures are related by Eq 272 & Eq 273. So, the bending effect in the thick-plate of thickness *t* measured in the z-direction is written:

$$\begin{Bmatrix} Mx \\ My \\ Mxy \end{Bmatrix} = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} \qquad \text{Eq 342}$$

Or,

$$\{M\} = [D]_{Bending}\{\chi\} \qquad \text{Eq 343}$$

So, since the transverse shear effect is considered in the Mindlin assumption, the shear strains are expressed by:

$$\gamma_{xz} = \theta_y + \frac{\partial w}{\partial x} \quad \Bigg| \quad \gamma_{yz} = -\theta_x + \frac{\partial w}{\partial y} \qquad \text{Eq 344}$$

As per the Theory of Elasticity, the shear effect in the thick plate is given by:

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} Gt\kappa & 0 \\ 0 & Gt\kappa \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \qquad \text{Eq 345}$$

Where G is the shear modulus given by:

$$G = \frac{E}{2(1+\nu)} \qquad \text{Eq 346}$$

In the Mindlin theory, the shear strain is assumed constant across the thickness of the plate. However, it is known to be parabolic across the thickness of the plate. To account for the inaccuracy in the shear strain, a shear correction factor κ is applied so that the correct amount of internal energy is predicted by the theory. The value of κ is usually taken to be $\pi^2/12$ or 5/6 (refer to a mechanic book). Eq 345 is also written:

$$\{Q\} = [D]_{Shear}\{\gamma\} \qquad \text{Eq 347}$$

So, the complete moment-shear-strain relationship for the thick-plate is:

$$\begin{Bmatrix} Mx \\ My \\ Mxy \\ Qx \\ Qy \end{Bmatrix} = \begin{bmatrix} \frac{Et^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \frac{Et}{2(1+\nu)}\begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \frac{\partial \theta_y}{\partial x} \\ -\frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x} \\ \theta_y + \frac{\partial w}{\partial x} \\ -\theta_x + \frac{\partial w}{\partial y} \end{Bmatrix} \qquad \text{Eq 348}$$

So,

$$\begin{Bmatrix} Mx \\ My \\ Mxy \\ Qx \\ Qy \end{Bmatrix} = \begin{bmatrix} \frac{Et^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \frac{Et}{2(1+\nu)}\begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \qquad \text{Eq 349}$$

Or,

$$\begin{Bmatrix} \{M\} \\ \{Q\} \end{Bmatrix} = \begin{bmatrix} [D]_{Bending} & [0] \\ [0] & [D]_{Shear} \end{bmatrix} \begin{Bmatrix} \{\chi\} \\ \{\gamma\} \end{Bmatrix} \qquad \text{Eq 350}$$

$$\begin{Bmatrix} \{M\} \\ \{Q\} \end{Bmatrix} = \begin{bmatrix} [D]_{Bending} & [0] \\ [0] & [D]_{Shear} \end{bmatrix} \{\varepsilon\} \qquad \text{Eq 351}$$

Where,

$$\{\varepsilon\} = \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = [B]\{\delta_i\} = [B]\begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} \qquad \text{Eq 352}$$

The constitutive matrix is then:

$$\begin{bmatrix} [D]_{Bending} & [0] \\ [0] & [D]_{Shear} \end{bmatrix} = \begin{bmatrix} \frac{Et^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \frac{Et}{2(1+\nu)}\begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix} \end{bmatrix} \qquad \text{Eq 353}$$

With:

- [B] the strain displacement matrix
- {δ} the nodal displacement vector

By substituting Eq 352 into Eq 351:

$$\begin{Bmatrix} \{M\} \\ \{Q\} \end{Bmatrix} = \begin{bmatrix} [D]_{Bending} & [0] \\ [0] & [D]_{Shear} \end{bmatrix} [B]\{\delta_i\} \qquad \text{Eq 354}$$

To evaluate [B], using Eq 339, Eq 341 and Eq 344. the curvatures and the shear strains are:

$$\chi_x = \sum_{i=1}^{4} \theta_{yi} \frac{\partial N_i}{\partial x} \qquad \chi_y = \sum_{i=1}^{4} -\theta_{xi} \frac{\partial N_i}{\partial y} \qquad \chi_{xy} = \sum_{i=1}^{4} \theta_{yi} \frac{\partial N_i}{\partial y} - \sum_{i=1}^{4} \theta_{xi} \frac{\partial N_i}{\partial x} \qquad \text{Eq 355}$$

$$\gamma_{xz} = \sum_{i=1}^{4} w_i \frac{\partial N_i}{\partial x} + \sum_{i=1}^{4} \theta_{yi} N_i \qquad \gamma_{yz} = \sum_{i=1}^{4} w_i \frac{\partial N_i}{\partial y} - \sum_{i=1}^{4} \theta_{xi} N_i$$

The Eq 355 written in matrix form is:

$$\begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & \frac{\partial N_i}{\partial x} \\ 0 & -\frac{\partial N_i}{\partial y} & 0 \\ 0 & -\frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial x} & 0 & N_i \\ \frac{\partial N_i}{\partial y} & -N_i & 0 \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} \qquad \text{Eq 356}$$

$$\{\varepsilon\} = \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = [B] \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix}$$

From Eq 356, we obtain the 5 x 3 strain displacement matrix [B]:

$$[B] = \begin{bmatrix} 0 & 0 & \frac{\partial N_i}{\partial x} \\ 0 & -\frac{\partial N_i}{\partial y} & 0 \\ 0 & -\frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial x} & 0 & N_i \\ \frac{\partial N_i}{\partial y} & -N_i & 0 \end{bmatrix} \qquad \text{Eq 357}$$

Then, for the four nodes quadrilateral element, using the nodal displacement vector from Eq 336, the strain displacement matrix becomes the following 5 x 12 matrix:

$$[B] = \begin{bmatrix} [B_j] & [B_k] & [B_m] & [B_n] \end{bmatrix} \qquad \text{Eq 358}$$

Where each $[B_i]$ for i = j, k, m, n is the 5 x 3 matrix given by Eq 357. [B] is then written as:

$$[B] = \begin{bmatrix} 0 & 0 & \frac{\partial N_j}{\partial x} & 0 & 0 & \frac{\partial N_k}{\partial x} & 0 & 0 & \frac{\partial N_m}{\partial x} & 0 & 0 & \frac{\partial N_n}{\partial x} \\ 0 & -\frac{\partial N_j}{\partial y} & 0 & 0 & -\frac{\partial N_k}{\partial y} & 0 & 0 & -\frac{\partial N_m}{\partial y} & 0 & 0 & -\frac{\partial N_n}{\partial y} & 0 \\ 0 & -\frac{\partial N_j}{\partial x} & \frac{\partial N_j}{\partial y} & 0 & -\frac{\partial N_k}{\partial x} & \frac{\partial N_k}{\partial y} & 0 & -\frac{\partial N_m}{\partial x} & \frac{\partial N_m}{\partial y} & 0 & -\frac{\partial N_n}{\partial x} & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_j}{\partial x} & 0 & N_j & \frac{\partial N_k}{\partial x} & 0 & N_k & \frac{\partial N_m}{\partial x} & 0 & N_m & \frac{\partial N_n}{\partial x} & 0 & N_n \\ \frac{\partial N_j}{\partial y} & -N_j & 0 & \frac{\partial N_k}{\partial y} & -N_k & 0 & \frac{\partial N_m}{\partial y} & -N_m & 0 & \frac{\partial N_n}{\partial y} & -N_n & 0 \end{bmatrix} \qquad \text{Eq 359}$$

So, using Eq 354, the bending and shear terms are:

$$\begin{bmatrix} [D]_{Bending} & [0] \\ [0] & [D]_{Shear} \end{bmatrix} [B] = \begin{bmatrix} [D]_{Bending} & [0] \\ [0] & [D]_{Shear} \end{bmatrix} \begin{bmatrix} [B_j] & [B_k] & [B_m] & [B_n] \end{bmatrix} \qquad \text{Eq 360}$$

For i = j, k, m, n, the ith submatrice in the Eq 360 is:

$$\begin{bmatrix} [D]_{Bending} & [0] \\ [0] & [D]_{Shear} \end{bmatrix} [B_i] = \frac{Et}{12(1+\nu)} \begin{bmatrix} 0 & \frac{-\nu t^2}{1-\nu}\frac{\partial N_i}{\partial y} & \frac{t^2}{1-\nu}\frac{\partial N_i}{\partial x} \\ 0 & \frac{-t^2}{1-\nu}\frac{\partial N_i}{\partial x} & \frac{\nu t^2}{1-\nu}\frac{\partial N_i}{\partial y} \\ 0 & \frac{-t^2}{2}\frac{\partial N_i}{\partial x} & \frac{t^2}{2}\frac{\partial N_i}{\partial y} \\ 6\kappa\frac{\partial N_i}{\partial x} & 0 & 6\kappa N_i \\ 6\kappa\frac{\partial N_i}{\partial y} & -6\kappa N_i & 0 \end{bmatrix} \qquad \text{Eq 361}$$

So, the bending and shear terms in Eq 361 can be separated and written as:

$$\begin{bmatrix} [D]_{Bending} & [0] \\ [0] & [D]_{Shear} \end{bmatrix} [B_i] =$$

$$= \frac{Et}{12(1+\nu)} \left[\begin{bmatrix} 0 & \frac{-\nu t^2}{1-\nu}\frac{\partial N_i}{\partial y} & \frac{t^2}{1-\nu}\frac{\partial N_i}{\partial x} \\ 0 & \frac{-t^2}{1-\nu}\frac{\partial N_i}{\partial x} & \frac{\nu t^2}{1-\nu}\frac{\partial N_i}{\partial y} \\ 0 & \frac{-t^2}{2}\frac{\partial N_i}{\partial x} & \frac{t^2}{2}\frac{\partial N_i}{\partial y} \end{bmatrix} + 6\kappa \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial N_i}{\partial x} & 0 & N_i \\ \frac{\partial N_i}{\partial y} & -N_i & 0 \end{bmatrix} \right]$$

Eq 362

<u>Element Stiffness Matrix</u>

The stiffness matrix of the four nodes thick-plate element is calculated using the usual form:

$$[K] = \iint [B]^T \begin{bmatrix} [D]_{Bending} & [0] \\ [0] & [D]_{Shear} \end{bmatrix} [B] dxdy \qquad \text{Eq 363}$$

Where $[B]^T[D]\{B\}$ is computed as:

$$[\bar{k}] = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} \\ 0 & -\frac{\partial N_i}{\partial y} & -\frac{\partial N_i}{\partial x} & 0 & -N_i \\ \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_i}{\partial y} & N_i & 0 \end{bmatrix}_{i=j,k,m,n}$$

$$\text{x} \begin{bmatrix} \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \frac{Et}{2(1+\nu)} \begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix} \end{bmatrix} \text{x} \begin{bmatrix} 0 & 0 & \frac{\partial N_i}{\partial x} \\ 0 & -\frac{\partial N_i}{\partial y} & 0 \\ 0 & -\frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial x} & 0 & N_i \\ \frac{\partial N_i}{\partial y} & -N_i & 0 \end{bmatrix}_{i=j,k,m,n}$$

Eq 364

So, $[\bar{k}]$ is a 12 x 12 matrix:

$$[\bar{k}] = [B]^T \begin{bmatrix} [D]_{Bending} & [0] \\ [0] & [D]_{Shear} \end{bmatrix} [B] = \begin{bmatrix} [\bar{k}_{11}] & [\bar{k}_{11}] & [\bar{k}_{13}] & [\bar{k}_{14}] \\ [\bar{k}_{21}] & [\bar{k}_{22}] & [\bar{k}_{23}] & [\bar{k}_{24}] \\ [\bar{k}_{31}] & [\bar{k}_{32}] & [\bar{k}_{33}] & [\bar{k}_{34}] \\ [\bar{k}_{41}] & [\bar{k}_{42}] & [\bar{k}_{43}] & [\bar{k}_{44}] \end{bmatrix}$$

Eq 365

Where each $[\bar{k}_{\alpha\beta}]$ is a 3 x 3 matrix defined as:

$$[\bar{k}_{\alpha\beta}] = \frac{Et}{12(1+\nu)}$$

$$X\begin{bmatrix} 6\kappa\left(\frac{\partial N_\alpha}{\partial x}\frac{\partial N_\beta}{\partial x}+\frac{\partial N_\alpha}{\partial y}\frac{\partial N_\beta}{\partial y}\right) & -6\kappa\frac{\partial N_\alpha}{\partial y}N_\beta & 6\kappa\frac{\partial N_\alpha}{\partial x}N_\beta \\ -6\kappa\frac{\partial N_\alpha}{\partial y}N_\beta & \left(\begin{array}{c}\frac{t^2}{1-\nu}\left(\frac{\partial N_\alpha}{\partial y}\frac{\partial N_\beta}{\partial y}\right)\\ +\frac{t^2}{2}\left(\frac{\partial N_\alpha}{\partial x}\frac{\partial N_\beta}{\partial x}\right)+6\kappa N_\alpha N_\beta\end{array}\right) & \left(\begin{array}{c}-\frac{\nu t^2}{1-\nu}\left(\frac{\partial N_\alpha}{\partial y}\frac{\partial N_\beta}{\partial x}\right)\\ -\frac{t^2}{2}\left(\frac{\partial N_\alpha}{\partial x}\frac{\partial N_\beta}{\partial y}\right)\end{array}\right) \\ 6\kappa\frac{\partial N_\alpha}{\partial x}N_\beta & \left(\begin{array}{c}-\frac{\nu t^2}{1-\nu}\left(\frac{\partial N_\alpha}{\partial x}\frac{\partial N_\beta}{\partial y}\right)\\ -\frac{t^2}{2}\left(\frac{\partial N_\alpha}{\partial y}\frac{\partial N_\beta}{\partial x}\right)\end{array}\right) & \left(\begin{array}{c}\frac{t^2}{1-\nu}\left(\frac{\partial N_\alpha}{\partial x}\frac{\partial N_\beta}{\partial x}\right)\\ +\frac{t^2}{2}\left(\frac{\partial N_\alpha}{\partial y}\frac{\partial N_\beta}{\partial y}\right)+6\kappa N_\alpha N_\beta\end{array}\right) \end{bmatrix}$$

Eq 366

So, the stiffness matrix of the four nodes thick-plate element is calculated as:

$$[K] = \iint [\bar{k}]dxdy \qquad \text{Eq 367}$$

To express the above equation in terms of the natural coordinates <ξ,η> in the isoparametric formulation, since [B] is a function of ξ and η, the integration with respect to ξ and η is possible using the determinant of the Jacobian matrix to transform the *x* and *y* variables into the <ξ,η> coordinates.

$$[K] = \int_{-1}^{+1}\int_{-1}^{+1}[\bar{k}]|J|d\xi d\eta \qquad \text{Eq 368}$$

The integral can be easily computed numerically by the FE solver.

5.5.7. Shell

Definition

The shell element can carry loads in almost all directions: it combines the membrane and plate elements capabilities. The shell element undergoes in-plane, bending and twisting deformations. It can be used to model large structures like the roof of a building, cylindrical and hemispherical tanks, fuselage of aircraft…

In order to derive the stiffness matrix of the shell element we can use the usual method presented for membranes and plates in previous paragraphs which consist in defining the shape functions, the strain-displacement and stress-strain relationships and then obtaining the element matrix. However, since the shell element combines the properties of the membrane and the plate elements previously discussed in detail, we will use a short cut method to derive the shell element stiffness matrix.

Degrees of Freedom

Let's consider the degrees of freedom of the membrane and the plate elements in a local element coordinate system:

Shell Element	=	Membrane Element	+	Plate Element
5 DOF's per node		2 DOF's per node		3 DOF's per node

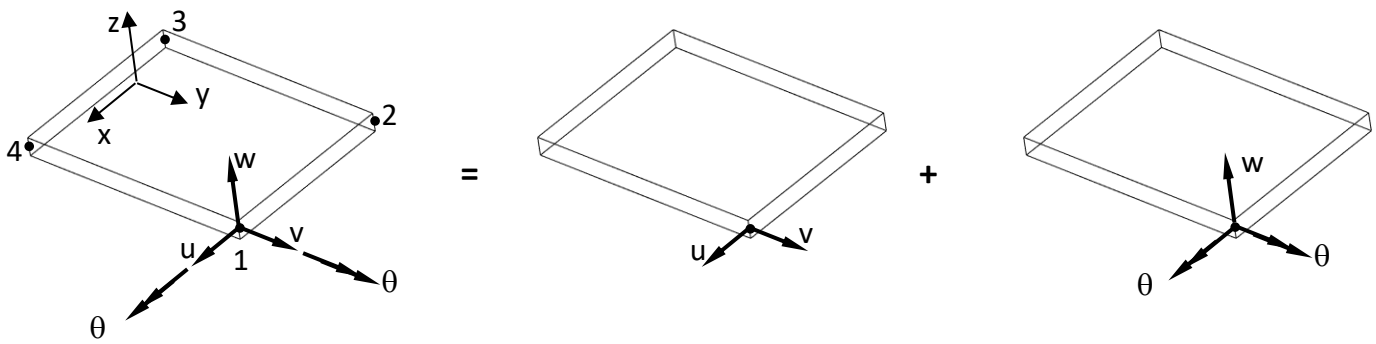


FIG 50 – DEGREES OF FREEDOM OF SHELL, MEMBRANE AND PLATE ELEMENT

The nodal displacement vector at node i (for i = 1, 2, 3, 4) is:

$$\{\delta_i\} = \begin{Bmatrix} u_i \\ v_i \\ w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} \qquad \text{Eq 369}$$

The total quadrilateral shell element displacement vector is composed with 5 x 4 = 20 degrees of freedom:

$$\{\delta\} = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} \qquad \text{Eq 370}$$

Element Stiffness Matrix

The stiffness matrix for the four nodes membrane element corresponding to the *u* and *v* DOF's is a 8 x 8 matrix:

$$[k_e]^{Membrane} = \begin{array}{c} \begin{array}{cccc} \text{Node 1} & \text{Node 2} & \text{Node 3} & \text{Node 4} \end{array} \\ \begin{bmatrix} k_{11}^m & k_{12}^m & k_{13}^m & k_{14}^m \\ k_{21}^m & k_{22}^m & k_{23}^m & k_{24}^m \\ k_{31}^m & k_{32}^m & k_{33}^m & k_{34}^m \\ k_{41}^m & k_{42}^m & k_{43}^m & k_{44}^m \end{bmatrix} \end{array} \qquad \text{Eq 371}$$

Where each submatrice $\left[k_{ij}^m\right]$ is a 2 x 2 membrane element matrix.

The stiffness matrix for the four nodes plate element corresponding to the *w*, θx and θy DOF's is a 12 x 12 matrix:

$$[k_e]^{Plate} = \begin{array}{c} \begin{array}{cccc} \text{Node 1} & \text{Node 2} & \text{Node 3} & \text{Node 4} \end{array} \\ \begin{bmatrix} k_{11}^p & k_{12}^p & k_{13}^p & k_{14}^p \\ k_{21}^p & k_{22}^p & k_{23}^p & k_{24}^p \\ k_{31}^p & k_{32}^p & k_{33}^p & k_{34}^p \\ k_{41}^p & k_{42}^p & k_{43}^p & k_{44}^p \end{bmatrix} \end{array} \qquad \text{Eq 372}$$

Where each submatrice $\left[k_{ij}^p\right]$ is a 3 x 3 plate element matrix. So, the stiffness matrix of the shell element is written as a 20 x 20 matrix:

$$[k_e]^{Shell} = \begin{bmatrix} [k_e]^{Membrane} & 0 \\ 0 & [k_e]^{Plate} \end{bmatrix} \qquad \text{Eq 373}$$

However, this 20 x 20 matrix is not convenient for the matrix transformation process which consists in transforming the matrix from the local coordinate system to the global coordinate system. So, an extended 24 x 24 matrix including the θz degree of freedom is used as shown below. Note that the stiffness corresponding to θz are zero. From a practical point of view, the stiffness corresponding to θz is set to a small value using a specific parameter for the shell elements, not a null value, to avoid singularities and obtain a solution.

$$[k_e]^{Shell} = \begin{array}{c} \begin{array}{cccccccccccc} \text{Node 1} & & & \text{Node 2} & & & \text{Node 3} & & & \text{Node 4} & & \end{array} \\ \begin{bmatrix} k_{11}^m & 0 & 0 & k_{12}^m & 0 & 0 & k_{13}^m & 0 & 0 & k_{14}^m & 0 & 0 \\ 0 & k_{11}^p & 0 & 0 & k_{12}^p & 0 & 0 & k_{13}^p & 0 & 0 & k_{14}^p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{21}^m & 0 & 0 & k_{22}^m & 0 & 0 & k_{23}^m & 0 & 0 & k_{24}^m & 0 & 0 \\ 0 & k_{21}^p & 0 & 0 & k_{22}^p & 0 & 0 & k_{23}^p & 0 & 0 & k_{24}^p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{31}^m & 0 & 0 & k_{32}^m & 0 & 0 & k_{33}^m & 0 & 0 & k_{34}^m & 0 & 0 \\ 0 & k_{31}^p & 0 & 0 & k_{32}^p & 0 & 0 & k_{33}^p & 0 & 0 & k_{34}^p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{41}^m & 0 & 0 & k_{42}^m & 0 & 0 & k_{43}^m & 0 & 0 & k_{44}^m & 0 & 0 \\ 0 & k_{41}^p & 0 & 0 & k_{42}^p & 0 & 0 & k_{43}^p & 0 & 0 & k_{43}^p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \text{Eq 374}$$

5.6. 3D SOLID ELEMENT

Let's consider the three-dimensional element, also called the solid element. This element is used to analyze general 3D bodies like dams, thick wall structures, solid forging parts like fittings or engine parts. As mentioned in chapter about "The Library of Elements" in "Practical Finite Element Analysis for Mechanical Engineers", the basic solid elements are the tetrahedron with 4 or 10 nodes. A more advanced solid element is the hexahedron, also called the brick element. In this paragraph, the stiffness matrix is derived for the 8 nodes and the 20 nodes hexahedron elements. The isoparametric formulation presented for a 2D element page 98 is used.

5.6.1. Linear Hexahedral Element

Degrees-of-freedom

The linear hexahedral element has eight corner nodes. Each node has three translational degrees of freedom *u*, *v*, *w*. So, the linear hexahedral element has 8 x 3 = 24 degrees of freedom.

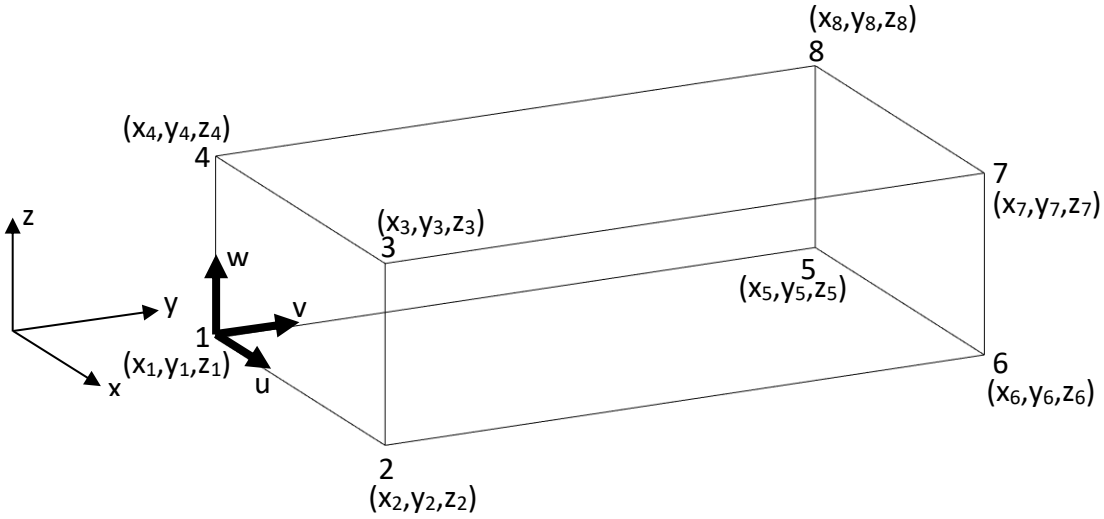


FIG 51 – DEGREES OF FREEDOM OF THE LINEAR HEXAHEDRAL ELEMENT

The nodal displacement vector at node i (for i = 1 to 8) is:

$$\{\delta_i\} = \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} \qquad \text{Eq 375}$$

The total linear hexahedral element displacement vector is composed with 8 x 3 = 24 degrees of freedom:

$$\{\delta\} = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \\ \delta_8 \end{Bmatrix} \qquad \text{Eq 376}$$

Shape Functions

The isoparametric natural coordinates <ξ,η,ψ> is used to define the shape functions and to facilitate the stiffness matrix derivation. The element faces are defined by ξ=±1, η=±1, ψ=±1 as shown in Fig 52.

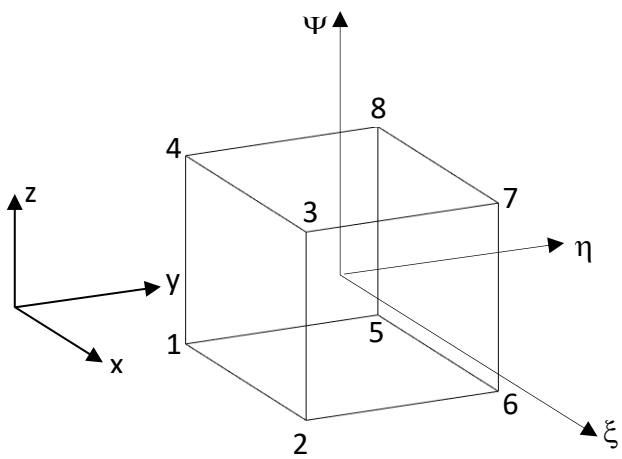


FIG 52 –LINEAR HEXAHEDRAL ELEMENT IN THE <ξ, η, Ψ> NATURAL COORDINATE SYSTEM

In the natural coordinate system <ξ,η,ψ>, the corner nodes have the following coordinates:

TABLE 3 – NODES NATURAL COORDINATES

Node	ξ_0	η_0	ψ_0
1	-1	-1	-1
2	1	-1	-1
3	1	-1	1
4	-1	-1	1
5	-1	1	-1
6	1	1	-1
7	1	1	1
8	-1	1	1

So, we need 24 coefficients to define the functions used to describe the element geometry for *x*, *y* and *z* in terms of the generalized degrees of freedom:

$$\begin{aligned} x &= a_1 + a_2\xi + a_3\eta + a_4\psi + a_5\xi\eta + a_6\eta\psi + a_7\psi\xi + a_8\xi\eta\psi \\ y &= a_9 + a_{10}\xi + a_{11}\eta + a_{12}\psi + a_{13}\xi\eta + a_{14}\eta\psi + a_{15}\psi\xi + a_{16}\xi\eta\psi \\ z &= a_{17} + a_{18}\xi + a_{19}\eta + a_{20}\psi + a_{21}\xi\eta + a_{22}\eta\psi + a_{23}\psi\xi \\ &\quad + a_{24}\xi\eta\psi \end{aligned} \qquad \text{Eq 377}$$

So, same as the Eq 304 extended to include the z coordinate:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{8} \left(\begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} \right) \qquad \text{Eq 378}$$

Where N_i are the shapes functions for i=1 to 8, with $\xi_0 = \pm 1$, $\eta_0 = \pm 1$, $\psi_0 = \pm 1$ as per Table 3:

$$N_i = \frac{1}{8}(1 + \xi\xi_0)(1 + \eta\eta_0)(1 + \psi\psi_0) \qquad \text{Eq 379}$$

The displacement functions in terms of the generalized degrees of freedom are of the same form as the functions used to describe the element geometry by Eq 377. So, the displacement functions including u, v, w is written:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^{8} \left(\begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} \right) \qquad \text{Eq 380}$$

The shape function matrix is a 3 x 24 matrix.

<u>Strain-Displacement & Stress-Strain Relationships</u>

Let's consider the three-dimensional infinitesimal element (dx,dy,dz) in a cartesian coordinate system as shown in Fig 53. It represents the state of stress on three perpendicular faces of a body.

- The normal stresses σ_X, σ_Y and σ_Z are perpendicular to the faces.
- The shear stresses act in the plane of each face: $\tau_{XY} = \tau_{YX}$, $\tau_{YZ} = \tau_{ZY}$, $\tau_{ZX} = \tau_{XZ}$

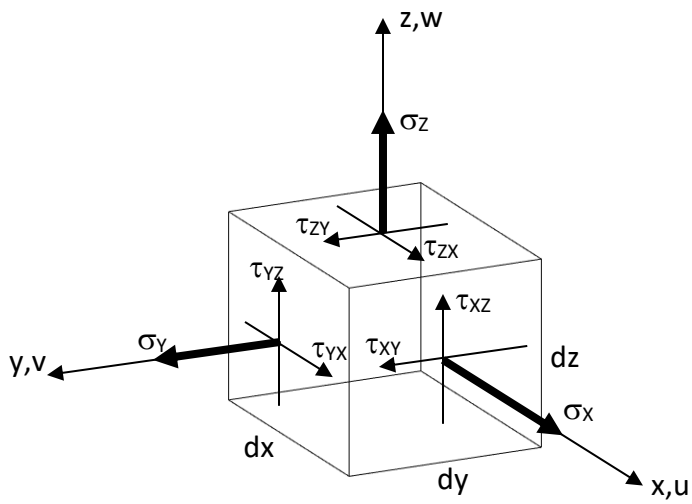


FIG 53 –STATE OF STRESS ON A SOLID ELEMENT

The element strain-displacement relationships are (*u, v, w* are the displacements associated with the *x, y, z* directions):

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad \Big| \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \Big| \quad \varepsilon_z = \frac{\partial w}{\partial z} \qquad \text{Eq 381}$$

The shear strains are:

$$\gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \Big| \quad \gamma_{yz} = \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad \Big| \quad \gamma_{zx} = \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \qquad \text{Eq 382}$$

The stress-strain relationships for an isotropic material are:

$$\{\sigma\} = [D]\{\varepsilon\} \qquad \text{Eq 383}$$

Where {σ} is the stress vector, {ε} is the strain vector and [D] is the constitutive matrix:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2} & 0 & 0 \\ & & & & \frac{1-2\nu}{2} & 0 \\ S & Y & M & & & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \qquad \text{Eq 384}$$

To define the strain-displacement matrix [B], Eq 381 & Eq 382 are used. These relations are expressed in terms of global coordinates, including a differentiation with respect to z. So, the Jacobian matrix presented for the isoparametric formulation of the four nodes quadrilateral element (see Eq 313) needs to be expanded:

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \psi} & \frac{\partial y}{\partial \psi} & \frac{\partial z}{\partial \psi} \end{bmatrix} \qquad \text{Eq 385}$$

So, Eq 312 is also expanded:

$$\frac{\partial j}{\partial x} = \frac{\begin{vmatrix} \frac{\partial j}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial j}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial j}{\partial \psi} & \frac{\partial y}{\partial \psi} & \frac{\partial z}{\partial \psi} \end{vmatrix}}{|J|} \qquad \frac{\partial j}{\partial y} = \frac{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial j}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial j}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \psi} & \frac{\partial j}{\partial \psi} & \frac{\partial z}{\partial \psi} \end{vmatrix}}{|J|} \qquad \frac{\partial j}{\partial z} = \frac{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial j}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial j}{\partial \eta} \\ \frac{\partial x}{\partial \psi} & \frac{\partial y}{\partial \psi} & \frac{\partial j}{\partial \psi} \end{vmatrix}}{|J|} \qquad \text{Eq 386}$$

So, from Eq 386:

$$\frac{\partial(\quad)}{\partial x} = \frac{1}{|J|}\left[\frac{\partial(\quad)}{\partial \xi}\left(\frac{\partial y}{\partial \eta}\frac{\partial z}{\partial \psi} - \frac{\partial y}{\partial \psi}\frac{\partial z}{\partial \eta}\right) - \frac{\partial y}{\partial \xi}\left(\frac{\partial(\quad)}{\partial \eta}\frac{\partial z}{\partial \psi} - \frac{\partial(\quad)}{\partial \psi}\frac{\partial z}{\partial \eta}\right) + \frac{\partial z}{\partial \xi}\left(\frac{\partial(\quad)}{\partial \eta}\frac{\partial y}{\partial \psi} - \frac{\partial(\quad)}{\partial \psi}\frac{\partial y}{\partial \eta}\right)\right] \qquad \text{Eq 387}$$

$$\frac{\partial(\quad)}{\partial y}=\frac{1}{|J|}\left[\frac{\partial x}{\partial \xi}\left(\frac{\partial(\quad)}{\partial \eta}\frac{\partial z}{\partial \psi}-\frac{\partial(\quad)}{\partial \psi}\frac{\partial z}{\partial \eta}\right)-\frac{\partial(\quad)}{\partial \xi}\left(\frac{\partial x}{\partial \eta}\frac{\partial z}{\partial \psi}-\frac{\partial x}{\partial \psi}\frac{\partial z}{\partial \eta}\right)+\frac{\partial z}{\partial \xi}\left(\frac{\partial x}{\partial \eta}\frac{\partial(\quad)}{\partial \psi}-\frac{\partial x}{\partial \psi}\frac{\partial(\quad)}{\partial \eta}\right)\right]$$

$$\frac{\partial(\quad)}{\partial z}=\frac{1}{|J|}\left[\frac{\partial x}{\partial \xi}\left(\frac{\partial y}{\partial \eta}\frac{\partial(\quad)}{\partial \psi}-\frac{\partial y}{\partial \psi}\frac{\partial(\quad)}{\partial \eta}\right)-\frac{\partial y}{\partial \xi}\left(\frac{\partial x}{\partial \eta}\frac{\partial(\quad)}{\partial \psi}-\frac{\partial x}{\partial \psi}\frac{\partial(\quad)}{\partial \eta}\right)+\frac{\partial(\quad)}{\partial \xi}\left(\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \psi}-\frac{\partial x}{\partial \psi}\frac{\partial y}{\partial \eta}\right)\right]$$

So, writing the strain-displacement equations Eq 381 & Eq 382 in matrix form:

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix}=\begin{bmatrix}\frac{\partial(\quad)}{\partial x} & 0 & 0\\ 0 & \frac{\partial(\quad)}{\partial y} & 0\\ 0 & 0 & \frac{\partial(\quad)}{\partial z}\\ \frac{\partial(\quad)}{\partial y} & \frac{\partial(\quad)}{\partial x} & 0\\ 0 & \frac{\partial(\quad)}{\partial z} & \frac{\partial(\quad)}{\partial y}\\ \frac{\partial(\quad)}{\partial z} & 0 & \frac{\partial(\quad)}{\partial x}\end{bmatrix}\begin{Bmatrix}u\\ v\\ w\end{Bmatrix} \qquad \text{Eq 388}$$

So, the operator matrix [Ψ] is defined by substituting Eq 387 into Eq 388 and the strains expressed in term of the natural coordinates <ξ,η,ψ> are obtained. Then, Eq 388 is expressed in terms of the shape functions in compact matrix form:

$$\{\varepsilon\}=[\Psi][N]\{\delta\} \qquad \text{Eq 389}$$

So, the strain-displacement matrix is given by:

$$[B]=[\Psi][N] \qquad \text{Eq 390}$$

Where,

- [Ψ] is the 6 x 3 operator matrix from Eq 388
- [N] is the 3 x 24 shape function matrix from Eq 379 & Eq 380

[B] is now a function of ξ,η and ψ. Note that [B] is of order 6 x 24.

<u>Element Stiffness Matrix</u>

The stiffness matrix of the linear hexahedral element is calculated using the usual equation:

$$[k]=\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1}[B]^T[D][B]|J|d\xi d\eta d\psi \qquad \text{Eq 391}$$

Where,

- [B] is computed by Eq 390
- [D] is computed by Eq 384
- [k] is a 24 x 24 matrix

Again, it is best to evaluate [k] by numerical integration using a FE solver.

5.6.2. Quadratic Hexahedral Element

Degrees of freedom

The quadratic hexahedral element has eight corner nodes and twelve mid-side nodes for a total of 20 nodes (see Fig 54). Like for the linear hexahedral element, each node has three translational degrees of freedom *u*, *v*, *w*. So, the linear hexahedral element has 20 x 3 = 60 degrees of freedom.

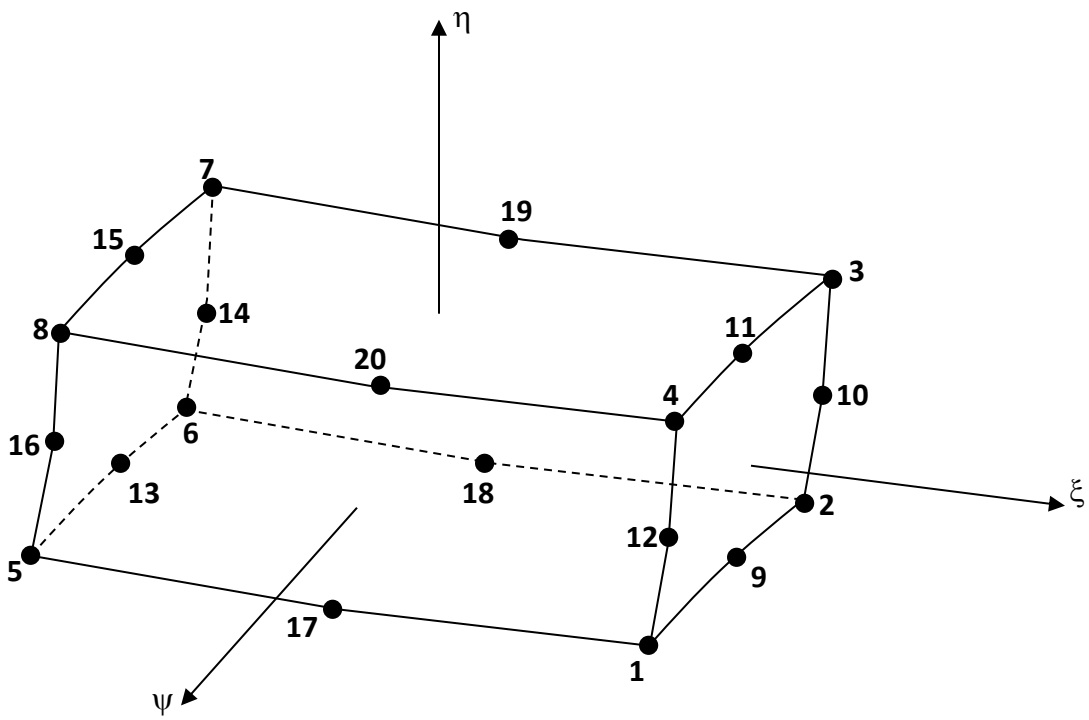


FIG 54 – DEGREES OF FREEDOM OF THE QUADRATIC HEXAHEDRAL ELEMENT

Shape Functions

So, a total of 60 coefficients are needed to define the functions used to describe the element geometry for *x*, *y* and *z* in terms of the generalized degrees of freedom:

$$\begin{aligned} x = a_1 &+ a_2\xi + a_3\eta + a_4\psi + a_5\xi\eta + a_6\eta\psi + a_7\psi\xi + a_8\xi^2 + a_9\eta^2 + a_{10}\psi^2 \\ &+ a_{11}\xi^2\eta + a_{12}\xi\eta^2 + a_{13}\eta^2\psi + a_{14}\eta\psi^2 + a_{15}\psi^2\xi + a_{16}\psi\xi^2 \\ &+ a_{17}\xi\eta\psi + a_{18}\xi^2\eta\psi + a_{19}\xi\eta^2\psi + a_{20}\xi\eta\psi^2 \end{aligned}$$

$$\begin{aligned} y = a_{21} &+ a_{22}\xi + a_{23}\eta + a_{24}\psi + a_{25}\xi\eta + a_{26}\eta\psi + a_{27}\psi\xi + a_{28}\xi^2 + a_{29}\eta^2 \\ &+ a_{30}\psi^2 \\ &+ a_{31}\xi^2\eta + a_{32}\xi\eta^2 + a_{33}\eta^2\psi + a_{34}\eta\psi^2 + a_{35}\psi^2\xi + a_{36}\psi\xi^2 \\ &+ a_{37}\xi\eta\psi + a_{38}\xi^2\eta\psi + a_{39}\xi\eta^2\psi + a_{40}\xi\eta\psi^2 \end{aligned} \qquad \text{Eq 392}$$

$$
\begin{aligned}
z = a_{41} + a_{42}\xi + a_{43}\eta + a_{44}\psi + a_{45}\xi\eta + a_{46}\eta\psi + a_{47}\psi\xi + a_{48}\xi^2 + a_{49}\eta^2 \\
+ a_{50}\psi^2 \\
+ a_{51}\xi^2\eta + a_{52}\xi\eta^2 + a_{53}\eta^2\psi + a_{54}\eta\psi^2 + a_{55}\psi^2\xi + a_{56}\psi\xi^2 \\
+ a_{57}\xi\eta\psi + a_{58}\xi^2\eta\psi + a_{59}\xi\eta^2\psi + a_{60}\xi\eta\psi^2
\end{aligned}
$$

In the natural coordinate system <ξ,η,ψ>, the corner and mid-side nodes have the following coordinates:

TABLE 4 – NODES NATURAL COORDINATES

Node	ξ_0	η_0	ψ_0
1	+1	-1	-1
2	+1	-1	-1
3	+1	+1	-1
4	+1	+1	+1
5	-1	-1	+1
6	-1	-1	-1
7	-1	+1	-1
8	-1	+1	+1
9	+1	-1	0
10	+1	0	-1
11	+1	+1	0
12	+1	0	+1
13	-1	-1	0
14	-1	0	-1
15	-1	+1	0
16	-1	0	+1
17	0	-1	+1
18	0	-1	-1
19	0	+1	-1
20	0	+1	+1

So, the shape functions are given by:

For the corner nodes
i = 1 to 8

$$N_i = \frac{1}{8}[(1+\xi\xi_0)(1+\eta\eta_0)(1+\psi\psi_0)][\xi\xi_0+\eta\eta_0+\psi\psi_0 - 2] \quad \text{Eq 393}$$

For the mid-side nodes
i = 9, 11, 13, 15

$$N_i = \frac{1}{4}[(1+\xi\xi_0)(1+\eta\eta_0)(1+\psi^2)] \quad \text{Eq 394}$$

For the mid-side nodes
i = 10, 12, 14, 16

$$N_i = \frac{1}{4}[(1+\xi\xi_0)(1+\eta^2)(1+\psi\psi_0)] \quad \text{Eq 395}$$

For the mid-side nodes
i = 17 to 20

$$N_i = \frac{1}{4}[(1+\xi^2)(1+\eta\eta_0)(1+\psi\psi_0)] \quad \text{Eq 396}$$

Element Stiffness Matrix

The derivation of the stiffness matrix for the quadratic hexahedral element follows the same steps presented for the linear hexahedral element except that the shape functions from Eq 393 to Eq 396 are used.

The stiffness matrix is computed numerically using Eq 391 and is a 60 x 60 matrix.

6. HOW IS THE STIFFNESS MATRIX ASSEMBLED?

When computing an FEA with software, the actual assembly of the stiffness matrix is invisible to the FEA analyst, but it is useful to know how this occurs.

The finite element model is a system composed of several elements connected at nodes. The global stiffness matrix of the model is built by assembling each individual element's stiffness matrix.

At the beginning of this chapter, the way the stiffness matrices of individual elements are computed, based on element properties (degrees of freedom, geometry, and materials), was presented. This section will explain how these elemental matrices are assembled into the global stiffness matrix for the entire model.

6.1. MATRIX ASSEMBLY

The assembly of elements consists of placing each elemental matrix in an appropriate position in the global matrix, based on degrees of freedom. The procedure is best understood through a simple example.

Let's consider a simple rectangle plate discretized using four constant strain triangular (CST) elements as shown in Fig 55. Six nodes are involved in this discretization. Each CST element has two degrees of freedom at each node, so the total number of degrees of freedom for the plate is 6 x 2 = 12. Then, the stiffness matrix for the plate meshed with four CST elements will have a size of 12 x 12. The elements and the nodes are numbered as shown in Fig 55.

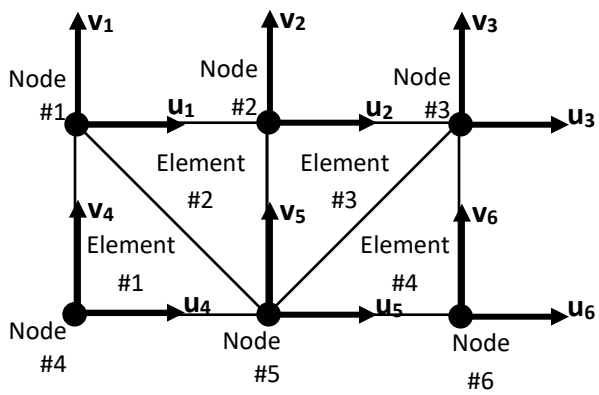


FIG 55 – SIMPLE RECTANGLE PLATE DISCRETIZED WITH FOUR CST ELEMENTS

The form of the stiffness matrix corresponding to the CST elements meshing is shown in Fig 56: two columns and two lines per element for a total of twelve degrees of freedom for the rectangular plate.

		Node 1		Node 2		Node 3		Node 4		Node 5		Node 6	
		u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	u_5	v_5	u_6	v_6
Node 1	u_1												
	v_1												
Node 2	u_2												
	v_2												
Node 3	u_3												
	v_3												
Node 4	u_4												
	v_4												
Node 5	u_5												
	v_5												
Node 6	u_6												
	v_6												

FIG 56 – STIFFNESS MATRIX FORM OF THE RECTANGLE PLATE MESHED WITH FOUR CST ELEMENTS

Let's closely examine how this global stiffness matrix is populated, which consists of passing the elements of the mesh one by one and identifying the degrees of freedom involved in the construction of the elemental stiffness matrix.

Element 1: The element #1 is connected at the nodes #1, #4 & #5. So, the degrees of freedom involved in the construction of the element #1 stiffness matrix are: (u_1,v_1), (u_4,v_4) & (u_5,v_5). Using the 6 x 6 stiffness matrix of the CST element derived in a previous paragraph (see Eq 238), we have the following matrix for the element #1 (note that the nodes are considered anti-clock wise; this approach will be used for all elements):

$$[k]_{Element\ \#1} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} \text{Node \#1} & & \text{Node \#4} & & \text{Node \#5} & \\ u_1 & v_1 & u_4 & v_4 & u_5 & v_5 \end{array} \\ \begin{bmatrix} \begin{bmatrix} k_{11x} & k_{11xy} \\ k_{11yx} & k_{11y} \end{bmatrix} & \begin{bmatrix} k_{14x} & k_{14xy} \\ k_{14yx} & k_{14y} \end{bmatrix} & \begin{bmatrix} k_{15x} & k_{15xy} \\ k_{15yx} & k_{15y} \end{bmatrix} \\ \begin{bmatrix} k_{41x} & k_{41xy} \\ k_{41yx} & k_{41y} \end{bmatrix} & \begin{bmatrix} k_{44x} & k_{44xy} \\ k_{44yx} & k_{44y} \end{bmatrix} & \begin{bmatrix} k_{45x} & k_{45xy} \\ k_{45yx} & k_{45y} \end{bmatrix} \\ \begin{bmatrix} k_{51x} & k_{51xy} \\ k_{51yx} & k_{51y} \end{bmatrix} & \begin{bmatrix} k_{54x} & k_{54xy} \\ k_{54yx} & k_{54y} \end{bmatrix} & \begin{bmatrix} k_{55x} & k_{55xy} \\ k_{55yx} & k_{55y} \end{bmatrix} \end{bmatrix} \end{array} \begin{array}{c} \\ \\ u_1 \\ v_1 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{array} \qquad \text{Eq 397}$$

Now, the key question is, at which location of the global stiffness matrix the element #1 matrix will be located? The answer lies in the list of degrees of freedom involved with the element #1 matrix formulation: (u_1,v_1), (u_4,v_4) & (u_5,v_5). The Fig 57 shows the elemental matrix Eq 397 written at the appropriate locations of the global stiffness matrix based on the degrees of freedom of the element #1.

		Node 1		Node 2		Node 3		Node 4		Node 5		Node 6	
		u_1	**v_1**	u_2	v_2	u_3	v_3	**u_4**	**v_4**	**u_5**	**v_5**	u_6	v_6
Node 1	**u_1**	k^{e1}_{11x}	k^{e1}_{11xy}					k^{e1}_{14x}	k^{e1}_{14xy}	k^{e1}_{15x}	k^{e1}_{15xy}		
	v_1	k^{e1}_{11yx}	k^{e1}_{11y}					k^{e1}_{14yx}	k^{e1}_{14y}	k^{e1}_{15yx}	k^{e1}_{15y}		
Node 2	u_2												
	v_2												
Node 3	u_3												
	v_3												
Node 4	**u_4**	k^{e1}_{41x}	k^{e1}_{41xy}					k^{e1}_{44x}	k^{e1}_{44xy}	k^{e1}_{45x}	k^{e1}_{45xy}		
	v_4	k^{e1}_{41yx}	k^{e1}_{41y}					k^{e1}_{44yx}	k^{e1}_{44y}	k^{e1}_{45yx}	k^{e1}_{45y}		
Node 5	**u_5**	k^{e1}_{51x}	k^{e1}_{51xy}					k^{e1}_{54x}	k^{e1}_{54xy}	k^{e1}_{55x}	k^{e1}_{55xy}		
	v_5	k^{e1}_{51yx}	k^{e1}_{51y}					k^{e1}_{54yx}	k^{e1}_{54y}	k^{e1}_{55yx}	k^{e1}_{55y}		
Node 6	u_6												
	v_6												

FIG 57 – GLOBAL STIFFNESS MATRIX POPULATED WITH ELEMENT #1 MATRIX

Element 2: The element #2 is connected at the nodes #1, #5 & #2 (again, note that the nodes are considered anti-clock wise). So, the degrees of freedom involved in the construction of the element #2 stiffness matrix are: (u_1,v_1), (u_5,v_5) & (u_2,v_2). Using the 6 x 6 stiffness matrix of the CST element derived in a previous paragraph (see Eq 238), we have the following matrix for the element #2:

$$[k]_{Element\ \#2} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} \text{Node \#1} & \text{Node \#5} & \text{Node \#2} \\ \begin{array}{cc} u_1 & v_1 \end{array} & \begin{array}{cc} u_5 & v_5 \end{array} & \begin{array}{cc} u_2 & v_2 \end{array} \end{array} \\ \begin{bmatrix} \begin{bmatrix} k_{11x} & k_{11xy} \\ k_{11yx} & k_{11y} \end{bmatrix} & \begin{bmatrix} k_{15x} & k_{15xy} \\ k_{15yx} & k_{15y} \end{bmatrix} & \begin{bmatrix} k_{12x} & k_{12xy} \\ k_{12yx} & k_{12y} \end{bmatrix} \\ \begin{bmatrix} k_{51x} & k_{51xy} \\ k_{51yx} & k_{51y} \end{bmatrix} & \begin{bmatrix} k_{55x} & k_{55xy} \\ k_{55yx} & k_{55y} \end{bmatrix} & \begin{bmatrix} k_{52x} & k_{52xy} \\ k_{52yx} & k_{52y} \end{bmatrix} \\ \begin{bmatrix} k_{21x} & k_{21xy} \\ k_{21yx} & k_{21y} \end{bmatrix} & \begin{bmatrix} k_{25x} & k_{25xy} \\ k_{25yx} & k_{25y} \end{bmatrix} & \begin{bmatrix} k_{22x} & k_{22xy} \\ k_{22yx} & k_{22y} \end{bmatrix} \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} u_1 \\ v_1 \\ u_5 \\ v_5 \\ u_2 \\ v_2 \end{array} \qquad \text{Eq 398}$$

The Fig 58 shows the elemental matrix Eq 398 written at the appropriate locations of the global stiffness matrix based on the degrees of freedom of the element #2.

		Node 1		Node 2		Node 3		Node 4		Node 5		Node 6	
		u_1	**v_1**	**u_2**	**v_2**	u_3	v_3	u_4	v_4	**u_5**	**v_5**	u_6	v_6
Node 1	**u_1**	k^{e2}_{11x}	k^{e2}_{11xy}	k^{e2}_{12x}	k^{e2}_{12xy}					k^{e2}_{15x}	k^{e2}_{15xy}		
	v_1	k^{e2}_{11yx}	k^{e2}_{11y}	k^{e2}_{12yx}	k^{e2}_{12y}					k^{e2}_{15yx}	k^{e2}_{15y}		
Node 2	**u_2**	k^{e2}_{21x}	k^{e2}_{21xy}	k^{e2}_{22x}	k^{e2}_{22xy}					k^{e2}_{25x}	k^{e2}_{25xy}		
	v_2	k^{e2}_{21yx}	k^{e2}_{21y}	k^{e2}_{22yx}	k^{e2}_{22y}					k^{e2}_{25yx}	k^{e2}_{25y}		
Node 3	u_3												
	v_3												
Node 4	u_4												
	v_4												
Node 5	**u_5**	k^{e2}_{51x}	k^{e2}_{51xy}	k^{e2}_{52x}	k^{e2}_{52xy}					k^{e2}_{55x}	k^{e2}_{55xy}		
	v_5	k^{e2}_{51yx}	k^{e2}_{51y}	k^{e2}_{52yx}	k^{e2}_{52y}					k^{e2}_{55yx}	k^{e2}_{55y}		
Node 6	u_6												
	v_6												

FIG 58 – GLOBAL STIFFNESS MATRIX POPULATED WITH ELEMENT #2 MATRIX

Element 3: The element #3 is connected at the nodes #2, #5 & #3. So, the degrees of freedom involved in the construction of the element #3 stiffness matrix are: (u_2,v_2), (u_5,v_5) & (u_3,v_3). Using the 6 x 6 stiffness matrix of the CST element derived in a previous paragraph (see Eq 238), we have the following matrix for the element #3:

$$[k]_{Element\ \#3} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccc} \text{Node \#2} & & \text{Node \#5} & & \text{Node \#3} & \\ u_2 & v_2 & u_5 & v_5 & u_3 & v_3 \end{array} \\ \begin{bmatrix} \begin{bmatrix} k_{22x} & k_{22xy} \\ k_{22yx} & k_{22y} \end{bmatrix} & \begin{bmatrix} k_{25x} & k_{25xy} \\ k_{25yx} & k_{25y} \end{bmatrix} & \begin{bmatrix} k_{23x} & k_{23xy} \\ k_{23yx} & k_{23y} \end{bmatrix} \\ \begin{bmatrix} k_{52x} & k_{52xy} \\ k_{52yx} & k_{52y} \end{bmatrix} & \begin{bmatrix} k_{55x} & k_{55xy} \\ k_{55yx} & k_{55y} \end{bmatrix} & \begin{bmatrix} k_{53x} & k_{53xy} \\ k_{53yx} & k_{53y} \end{bmatrix} \\ \begin{bmatrix} k_{32x} & k_{32xy} \\ k_{32yx} & k_{32y} \end{bmatrix} & \begin{bmatrix} k_{35x} & k_{35xy} \\ k_{35yx} & k_{35y} \end{bmatrix} & \begin{bmatrix} k_{33x} & k_{33xy} \\ k_{33yx} & k_{33y} \end{bmatrix} \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} u_2 \\ v_2 \\ u_5 \\ v_5 \\ u_3 \\ v_3 \end{array} \qquad \text{Eq 399}$$

The Fig 58 shows the elemental matrix Eq 399 written at the appropriate locations of the global stiffness matrix based on the degrees of freedom of the element #3.

		Node 1		Node 2		Node 3		Node 4		Node 5		Node 6	
		u_1	v_1	**u_2**	**v_2**	**u_3**	**v_3**	u_4	v_4	**u_5**	**v_5**	u_6	v_6
Node 1	u_1												
	v_1												
Node 2	**u_2**			k^{e3}_{22x}	k^{e3}_{22xy}	k^{e3}_{23x}	k^{e3}_{23xy}			k^{e3}_{25x}	k^{e3}_{25xy}		
	v_2			k^{e3}_{22yx}	k^{e3}_{22y}	k^{e3}_{23yx}	k^{e3}_{23y}			k^{e3}_{25yx}	k^{e3}_{25y}		
Node 3	**u_3**			k^{e3}_{32x}	k^{e3}_{32xy}	k^{e3}_{33x}	k^{e3}_{33xy}			k^{e3}_{35x}	k^{e3}_{35xy}		
	v_3			k^{e3}_{32yx}	k^{e3}_{32y}	k^{e3}_{33yx}	k^{e3}_{33y}			k^{e3}_{35yx}	k^{e3}_{35y}		
Node 4	u_4												
	v_4												
Node 5	**u_5**			k^{e3}_{52x}	k^{e3}_{52xy}	k^{e3}_{53x}	k^{e3}_{53xy}			k^{e3}_{55x}	k^{e3}_{55xy}		
	v_5			k^{e3}_{52yx}	k^{e3}_{52y}	k^{e3}_{53yx}	k^{e3}_{53y}			k^{e3}_{55yx}	k^{e3}_{55y}		
Node 6	u_6												
	v_6												

FIG 59 – GLOBAL STIFFNESS MATRIX POPULATED WITH ELEMENT #3 MATRIX

Element 4: The element #4 is connected at the nodes #3, #5 & #6. So, the degrees of freedom involved in the construction of the element #4 stiffness matrix are: (u_3,v_3), (u_5,v_5) & (u_6,v_6). Using the 6 x 6 stiffness matrix of the CST element derived in a previous paragraph (see Eq 238), we have the following matrix for the element #4:

$$[k]_{Element\ \#4} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} \text{Node \#3} & \text{Node \#5} & \text{Node \#6} \\ \begin{array}{cc} u_3 & v_3 \end{array} & \begin{array}{cc} u_5 & v_5 \end{array} & \begin{array}{cc} u_6 & v_6 \end{array} \end{array} \\ \begin{bmatrix} \begin{bmatrix} k_{33x} & k_{33xy} \\ k_{33yx} & k_{33y} \end{bmatrix} & \begin{bmatrix} k_{35x} & k_{35xy} \\ k_{35yx} & k_{35y} \end{bmatrix} & \begin{bmatrix} k_{36x} & k_{36xy} \\ k_{36yx} & k_{36y} \end{bmatrix} \\ \begin{bmatrix} k_{53x} & k_{53xy} \\ k_{53yx} & k_{53y} \end{bmatrix} & \begin{bmatrix} k_{55x} & k_{55xy} \\ k_{55yx} & k_{55y} \end{bmatrix} & \begin{bmatrix} k_{56x} & k_{56xy} \\ k_{56yx} & k_{56y} \end{bmatrix} \\ \begin{bmatrix} k_{63x} & k_{63xy} \\ k_{63yx} & k_{63y} \end{bmatrix} & \begin{bmatrix} k_{65x} & k_{65xy} \\ k_{65yx} & k_{65y} \end{bmatrix} & \begin{bmatrix} k_{66x} & k_{66xy} \\ k_{66yx} & k_{66y} \end{bmatrix} \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} u_3 \\ v_3 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{array} \qquad \text{Eq 400}$$

The Fig 60 shows the elemental matrix Eq 400 written at the appropriate locations of the global stiffness matrix based on the degrees of freedom of the element #4.

		Node 1		Node 2		Node 3		Node 4		Node 5		Node 6	
		u_1	v_1	u_2	v_2	**u_3**	**v_3**	u_4	v_4	**u_5**	**v_5**	**u_6**	**v_6**
Node 1	u_1												
	v_1												
Node 2	u_2												
	v_2												
Node 3	**u_3**					k_{33x}^{e4}	k_{33xy}^{e4}			k_{35x}^{e4}	k_{35xy}^{e4}	k_{36x}^{e4}	k_{36xy}^{e4}
	v_3					k_{33yx}^{e4}	k_{33y}^{e4}			k_{35yx}^{e4}	k_{35y}^{e4}	k_{36yx}^{e4}	k_{36y}^{e4}
Node 4	u_4												
	v_4												
Node 5	**u_5**					k_{53x}^{e4}	k_{53xy}^{e4}			k_{55x}^{e4}	k_{55xy}^{e4}	k_{56x}^{e4}	k_{56xy}^{e4}
	v_5					k_{53yx}^{e4}	k_{53y}^{e4}			k_{55yx}^{e4}	k_{55y}^{e4}	k_{56yx}^{e4}	k_{56y}^{e4}
Node 6	**u_6**					k_{63x}^{e4}	k_{63xy}^{e4}			k_{65x}^{e4}	k_{65xy}^{e4}	k_{66x}^{e4}	k_{66xy}^{e4}
	v_6					k_{63yx}^{e4}	k_{63y}^{e4}			k_{65yx}^{e4}	k_{65y}^{e4}	k_{66yx}^{e4}	k_{66y}^{e4}

FIG 60 – GLOBAL STIFFNESS MATRIX POPULATED WITH ELEMENT #4 MATRIX

Fig 61 shows a summary of the elements with their associated degrees of freedom (u_i,v_i) for i = 1 to 6:

		Node 1		Node 2		Node 3		Node 4		Node 5		Node 6	
		u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	u_5	v_5	u_6	v_6
Node 1	u_1	E1,E2		E2		0	0	E1		E1,E2		0	0
	v_1					0	0					0	0
Node 2	u_2	E2		E2,E3		E3		0	0	E2,E3		0	0
	v_2							0	0			0	0
Node 3	u_3	0	0	E3		E3,E4		0	0	**E3,E4**		E4	
	v_3	0	0					0	0				
Node 4	u_4	E1		0	0	0	0	E1		E1		0	0
	v_4			0	0	0	0					0	0
Node 5	u_5	E1,E2		E2,E3		E3,E4		E1		E1,E2,E3,E4		E4	
	v_5												
Node 6	u_6	0	0	0	0	E4		0	0	E4		E4	
	v_6	0	0	0	0			0	0				

FIG 61 – GLOBAL STIFFNESS MATRIX POPULATED WITH FOUR ELEMENTS OF THE DISCRETIZATION

The global structure stiffness matrix [K] for the rectangle plate meshed with four constant strain triangular elements is obtained using the direct stiffness method:

$$[K] = \sum_{e=1}^{4} [k]_e \qquad \text{Eq 401}$$

Where $[k]_e$ is the stiffness matrix of each element of the discretization. For example, for the position framed in bold in Fig 61, the involved degrees of freedom are (u_3,v_3) and (u_5,v_5) for the elements #3 and #4. So the coefficients of the matrix at this position are:

$$\begin{bmatrix} k_{35x} & k_{35xy} \\ k_{35yx} & k_{35y} \end{bmatrix}^{Element\ \#3} + \begin{bmatrix} k_{35x} & k_{35xy} \\ k_{35yx} & k_{35y} \end{bmatrix}^{Element\ \#4} \qquad \text{Eq 402}$$

$$\begin{bmatrix} k_{35x}^{e3} + k_{35x}^{e4} & k_{35xy}^{e3} + k_{35xy}^{e4} \\ k_{35yx}^{e3} + k_{35yx}^{e4} & k_{35y}^{e3} + k_{35y}^{e4} \end{bmatrix} \qquad \text{Eq 403}$$

Fig 61 shows that half of the positions in the matrix have null values. Moreover Fig 61 shows the symmetry of the stiffness matrix. So, only the upper portion of the stiffness matrix needs to be computed and stored to perform the structural analysis.

6.2. TAKING ADVANTAGE OF SPARSITY AND SYMMETRY

The global stiffness matrix for a structural FEM is a square matrix having the size (Total Number of degrees of freedom) x (Total Number of degrees of freedom). In a complex medium- or large-sized structural analysis, it is common to have a few million degrees of freedom, in total.

Let's consider a small model with 100,000 degrees of freedom. The number of entries in the stiffness matrix will be 100,000 x 100,000 = 10^{10}. Therefore, assuming that each entry is saved in a double precision variable of eight bytes, the memory required to store this matrix is 80 gigabytes. Manipulating a global stiffness matrix of this size is not possible.

However, there are two properties of the global stiffness matrix that can be exploited to reduce its size in memory:

- Matrix sparsity
- Matrix symmetry

In structural analysis, due to the method used to connect elements, many stiffness matrix [K] entries are null. We call such a stiffness matrix *sparse*. Since a zero value does not contribute to the solution, resolution algorithms ignore the null terms, and they do not need to be stored. Another interesting property of the stiffness matrix is its symmetry: the upper terms (above the diagonal) are identical to the lower terms (below the diagonal). Therefore, only half of the stiffness matrix must be stored—usually, the diagonal and upper terms. These two fundamental properties of the global stiffness matrix reduce the required memory and allow the FEA software to use smart schemes for storing the stiffness matrix, generally in one of the following two formats:

- Banded Matrix
- Skyline Matrix Storage

6.3. BANDED MATRIX

In matrix theory, a banded matrix is a sparse matrix whose non-zero entries are confined to a diagonal band, comprising the main diagonal and zero or more diagonals on either side. Fig 5 41 illustrates the banded matrix concept. Each black dot represents a term of the stiffness matrix. The left view shows the matrix before bandwidth optimization. The right view shows the same matrix with an optimized bandwidth.

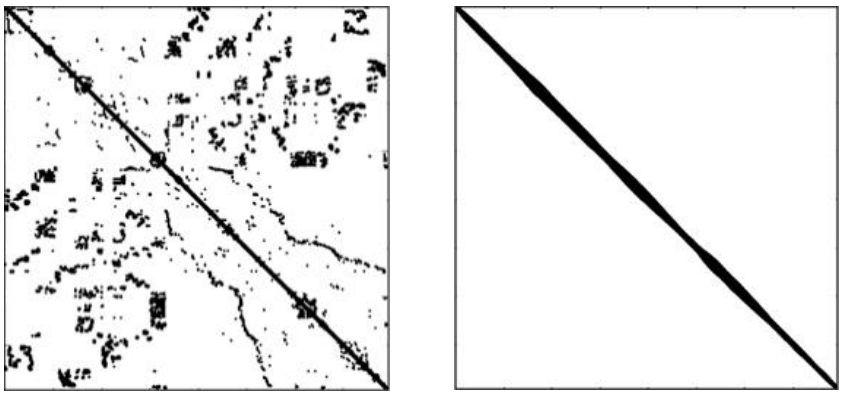


FIG 62 – (LEFT) ORIGINAL SPARE MATRIX (RIGHT) MATRIX WITH OPTIMIZED BANDWIDTH

Banded matrices are common in many fields of scientific computing that require the solving of large systems. A banded matrix has non-zero values only above and below the diagonal. From a computational perspective, banded matrices are always preferred to similarly dimensioned square matrices, because the work involved in performing operations such as multiplication decreases significantly, which greatly reduces computation time. One of the most efficient algorithms for minimizing bandwidth is the Cuthill–McKee algorithm, named after Elizabeth Cuthill and James McKee. This algorithm uses a method based on renumbering the nodes that constitute the mesh.

From a practical point of view, the maximum bandwidth for a given matrix can be found by considering each element in a mesh and finding the element with the largest difference in node ID. To understand the concept, let's consider the following simple example of a truss structure composed of six elements and eight nodes, with node numbering as shown in Fig 63.

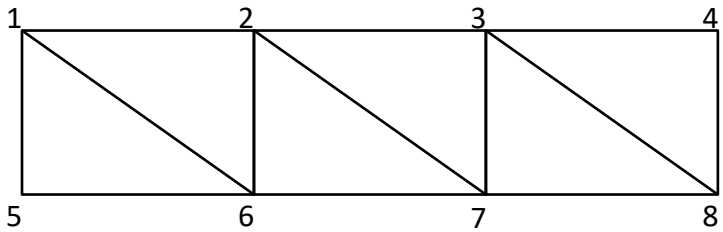


FIG 63 – ORIGINAL TRUSS STRUCTURE BEFORE OPTIMIZATION

The stiffness matrix for this node arrangement is (the non-zero terms are denoted by a *X* and the blank terms are null):

$$[K_{Original}] = \begin{bmatrix} X & X & & & X & X & & \\ X & X & X & & & X & X & \\ & X & X & X & & & X & \\ & & X & X & & & & X \\ X & & & & X & X & & \\ X & X & & & X & X & X & \\ & X & & & & X & X & X \\ & & & X & & & X & X \end{bmatrix} \qquad \text{Eq 404}$$

Looking at the matrix above, it is noted that the non-zero terms extend five columns beyond the main diagonal. It is possible to reduce the bandwidth by numbering the nodes differently. So, let's renumber the nodes as follow:

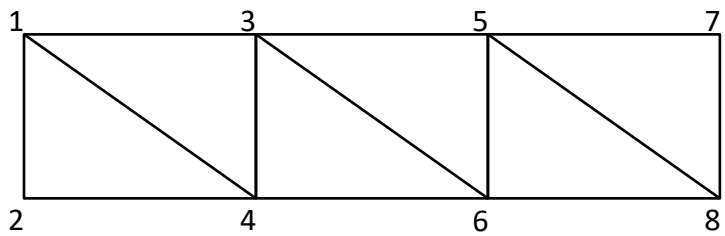


FIG 64 – NODES NUMBERING FOR BANDWIDTH OPTIMIZATION

The new stiffness matrix becomes:

$$[K_{Optimized}] = \begin{bmatrix} X & X & X & X & & & & \\ X & X & & X & & & & \\ X & & X & X & X & X & & \\ X & X & X & X & & X & & \\ & & X & & X & X & X & X \\ & & X & X & X & X & & X \\ & & & & X & & X & X \\ & & & & X & X & X & X \end{bmatrix} \qquad \text{Eq 405}$$

The new node numbering gives a reduced bandwidth with three columns beyond the main diagonal. In the commercial FEA codes, this operation is automatically performed without any action from the user being required.

6.4. SKYLINE MATRIX STORAGE

A symmetric sparse matrix can be stored using a matrix format called skyline matrix storage. This format significantly reduces storage space and requires less computation.

To understand the skyline storage format, let's look at another simple example. Consider the following N x N = 6 x 6 sparse stiffness matrix:

$$[K] = \begin{bmatrix} K_{11} & 0 & K_{13} & 0 & 0 & K_{16} \\ & K_{22} & 0 & K_{24} & 0 & 0 \\ & & K_{33} & K_{34} & 0 & 0 \\ & & & K_{44} & 0 & K_{46} \\ & & & & K_{55} & K_{56} \\ s & y & m & & & K_{66} \end{bmatrix} \qquad \text{Eq 406}$$

Since the stiffness matrix is symmetric, only the upper portion is considered. The envelope of *[K]* is defined as follows: from each diagonal entry move up the corresponding column until the last non-zero entry is found. The remaining zero entries are removed. This envelope, called the skyline template, defines the skyline profile of the matrix. A sparse matrix that can be profitably stored in this form is called a skymatrix. Note that the skyline template may include zero entries.

$$[K] = \begin{bmatrix} K_{11} & 0 & K_{13} & 0 & 0 & K_{16} \\ & K_{22} & 0 & K_{24} & 0 & 0 \\ & & K_{33} & K_{34} & 0 & 0 \\ & & & K_{44} & 0 & K_{46} \\ & & & & K_{55} & K_{56} \\ s & y & m & & & K_{66} \end{bmatrix} \longleftarrow \text{Skyline template}$$

Only entries in the skyline template must be stored, since null terms that are outside it will not influence the results. A one-dimensional array is used to store the terms of the skyline template:

$$s : [K_{11}, K_{22}, K_{13}, 0, K_{33}, K_{24}, K_{34}, K_{44}, K_{55}, K_{16}, 0, 0, K_{46}, K_{56}, K_{66}] \qquad \text{Eq 407}$$

The skyline array Eq 407 is complemented with an integer array *p* containing the addresses of the diagonal terms in the skyline array *s*. The array *p* has (N+1) entries. The $(i+1)^{th}$ entry of *p* contains the location of ith diagonal entry of [K] in *s*. So, for the example matrix:

$$p : [0, 1, 2, 5, 8, 9, 15] \qquad \text{Eq 408}$$

The terms of *p* for which the displacement component is prescribed are identified by a negative diagonal location value in the *p* array. For example, if u_4 and u_6 are the prescribed displacement components in the example, then:

$$p : [0, 1, 2, 5, -8, 9, -15] \qquad \text{Eq 409}$$

So, finally, the symmetric squared stiffness matrix Eq 406 originally containing 36 terms is stored in the following reduced format:

$$S = \{p , s\}$$

$$S = \{[0, 1, 2, 5, -8, 9, -15] , [K_{11}, K_{22}, K_{13}, 0, K_{33}, K_{24}, K_{34}, K_{44}, K_{55}, K_{16}, 0, 0, K_{46}, K_{56}, K_{66}]\} \qquad \text{Eq 410}$$

7. HOW ARE FEM EQUATIONS SOLVED?

A finite element analysis software is a set of algorithms, whose main function is to solve equilibrium equations written in matrix form. Computational techniques and hardware evolve so quickly that FEA software are frequently updated. The user can choose from a large library of resolution methods to solve a problem. Among them, two main techniques stand out:

- The direct method, based on LDL^T decomposition
- The iterative method, based on substitution of an assumed solution into a system of equations to obtain a better estimate of an unknown

7.1. DIRECT SOLUTION

With the direct method, the equilibrium equation *{P} = [K]{u}* is solved for unknown displacements *{u}*, without inverting *[K]*. LDL^T decomposition, the matrix form of Gaussian elimination, is an efficient and accurate method, comprised of three steps:

- Step 1: Factorization: $[K] = [L].[D].[L]^T$
- Step 2: Forward Substitution: $\{P\} = [L].\{y\}$
- Step 3: Backward Substitution: $\{y\} = [D].[L]^T.\{u\}$

Step 1: Factorization

The stiffness matrix [K] is replaced by the product of three matrices [L], [D] and $[L]^T$:

$$[K] = [L].[D].[L]^T \qquad \text{Eq 411}$$

[L] is the unit lower triangular matrix
[D] is the diagonal matrix
[L] is the unit upper triangular matrix

The factorization concept is illustrated for a n x n matrix:

$$\begin{bmatrix} \boldsymbol{K_{11}} & K_{12} & K_{13} & K_{14} & . & . & K_{1n} \\ K_{12} & \boldsymbol{K_{22}} & K_{23} & K_{24} & . & . & K_{2n} \\ K_{13} & K_{23} & \boldsymbol{K_{33}} & K_{34} & . & . & K_{3n} \\ K_{14} & K_{24} & K_{34} & \boldsymbol{K_{44}} & . & . & K_{4n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ K_{1n} & K_{2n} & K_{3n} & K_{4n} & . & . & \boldsymbol{K_{nn}} \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & . & . & 0 \\ L_{21} & \mathbf{1} & 0 & 0 & . & . & 0 \\ L_{31} & L_{32} & \mathbf{1} & 0 & . & . & 0 \\ L_{41} & L_{42} & L_{43} & \mathbf{1} & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ L_{n1} & L_{n2} & L_{n3} & L_{n4} & . & . & \mathbf{1} \end{bmatrix} \begin{bmatrix} D_{11} & 0 & 0 & 0 & . & . & 0 \\ 0 & D_{22} & 0 & 0 & . & . & 0 \\ 0 & 0 & D_{33} & 0 & . & . & 0 \\ 0 & 0 & 0 & D_{44} & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & D_{nn} \end{bmatrix} \begin{bmatrix} \mathbf{1} & L_{21} & L_{31} & L_{41} & . & . & L_{n1} \\ 0 & \mathbf{1} & L_{32} & L_{42} & . & . & L_{n2} \\ 0 & 0 & \mathbf{1} & L_{43} & . & . & L_{n3} \\ 0 & 0 & 0 & \mathbf{1} & . & . & L_{n4} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & \mathbf{1} \end{bmatrix}$$

Eq 412

The decomposition means that the stiffness matrix [K] is separated into the product of three simple matrices. The decomposition process consists in finding the factors for [D] and [L].

To compute the factors of the decomposition, the Cholesky decomposition is used. This method is very efficient in terms of memory storage capacity, computational cost and speed. The factors of the Cholesky decomposition are computed as:

$$D_{jj} = K_{jj} - \sum_{k=1}^{j-1} L_{jk}^2 D_{kk} \qquad \text{Eq 413}$$

For i > j

$$L_{ij} = \frac{1}{D_{jj}} \left(K_{ij} - \sum_{k=1}^{j-1} L_{ik} L_{jk} D_{kk} \right) \qquad \text{Eq 414}$$

The implementation of the Cholesky decomposition is:

- The subscript *i* is the row index and the subscript *j* is the column index
- The decomposition is computed column by column
- The decomposition starts by the evaluation of the diagonal term of the column using Eq 413
- The other elements are evaluated row by row using Eq 414

<u>Step 2: Forward Substitution</u>

From the factorization performed at step 1, the equilibrium equation can be written:

$$[P] = [K].\{u\} = [L].[D].[L]^T.\{u\} = [L].\{y\} \qquad \text{Eq 415}$$

Where,

$$\{y\} = [D].[L]^T.\{u\} \qquad \text{Eq 416}$$

So {y} is computed by starting with the first equation in the system:

$$[P] = [L].\{y\}$$

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ . \\ . \\ f_n \end{Bmatrix} = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & . & . & 0 \\ L_{21} & \mathbf{1} & 0 & 0 & . & . & 0 \\ L_{31} & L_{32} & \mathbf{1} & 0 & . & . & 0 \\ L_{41} & L_{42} & L_{43} & \mathbf{1} & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ L_{n1} & L_{n2} & L_{n3} & L_{n4} & . & . & \mathbf{1} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ . \\ . \\ y_n \end{Bmatrix} \qquad \text{Eq 417}$$

So, starting from the first equation:

$$f_1 = y_1$$

And continuing with the next lines using the results obtained from the previous lines:

$f_2 = L_{21}.y_1 + y_2 \Rightarrow y_2 = f_2 - L_{21}.y_1$

$f_3 = L_{31}.y_1 + L_{32}.y_2 + y_3 \Rightarrow y_3 = f_3 - L_{31}.y_1 - L_{32}.y_2$

…and so on.

So, the solution for {y} is given by:

$$y_i = f_i - \sum_{j=1}^{i-1} L_{ij}.y_j \qquad \text{For i = 2 to n} \qquad \text{Eq 418}$$

Step 3: Backward Substitution

Having computed {y} at step 2, it is easy to compute the unknown displacements {u} from {y} = [D].[L]^T.{u} by starting with the last equation:

$$\{y\} = [D].[L]^T.\{u\}$$

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ . \\ . \\ y_n \end{Bmatrix} = \begin{bmatrix} D_{11} & 0 & 0 & 0 & . & . & 0 \\ 0 & D_{22} & 0 & 0 & . & . & 0 \\ 0 & 0 & D_{33} & 0 & . & . & 0 \\ 0 & 0 & 0 & D_{44} & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & D_{nn} \end{bmatrix} \begin{bmatrix} \mathbf{1} & L_{21} & L_{31} & L_{41} & . & . & L_{n1} \\ 0 & \mathbf{1} & L_{32} & L_{42} & . & . & L_{n2} \\ 0 & 0 & \mathbf{1} & L_{43} & . & . & L_{n3} \\ 0 & 0 & 0 & \mathbf{1} & . & . & L_{n4} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & \mathbf{1} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ . \\ . \\ u_n \end{Bmatrix} \qquad \text{Eq 419}$$

Starting from the last equation, i=n:

$$y_n = D_{nn}.u_n \qquad \text{Eq 420}$$

u_n is easily found,

$$u_n = \frac{y_n}{D_{nn}} \qquad \text{Eq 421}$$

Then, moving backward in the Eq 419, {u} is solved by:

$$u_i = \frac{1}{D_{ii}}\left(y_i - \sum_{j=i+1}^{N} L_{ji}u_j\right) \qquad \text{For i = (n-1) to 1} \qquad \text{Eq 422}$$

7.2. ITERATIVE SOLUTION

An alternative to the direct method is an iterative method used to solve a linear system of equations, similar to the Jacobi method. Called the Gauss–Seidel method for the German mathematicians Carl Friedrich Gauss and Philipp Ludwig von Seidel, it is also known as the Liebmann method, or the method of successive displacement. Though it can be applied to any matrix with non-zero elements on the diagonals, convergence is only guaranteed if the matrix is either diagonally dominant, or symmetric and positive definite. The Gauss-Seidel method can be used to solve a square system of *n* equations with unknown displacements *{u}*:

{P} = [K]{u}

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ . \\ . \\ f_n \end{Bmatrix} = \begin{bmatrix} \boldsymbol{K_{11}} & K_{12} & K_{13} & K_{14} & . & . & K_{1n} \\ K_{21} & \boldsymbol{K_{22}} & K_{23} & K_{24} & . & . & K_{2n} \\ K_{31} & K_{32} & \boldsymbol{K_{33}} & K_{34} & . & . & K_{3n} \\ K_{41} & K_{42} & K_{43} & \boldsymbol{K_{44}} & . & . & K_{4n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ K_{n1} & K_{n2} & K_{n3} & K_{n4} & . & . & \boldsymbol{K_{nn}} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ . \\ . \\ u_n \end{Bmatrix} \qquad \text{Eq 423}$$

First, the stiffness matrix [K] is decomposed into a lower triangular component $[L]_{Lower}$ and an upper triangular component $[L]^{Upper}$:

$[K] = [L]_{Lower} + [L]^{Upper}$

$$\begin{bmatrix} \boldsymbol{K_{11}} & K_{12} & K_{13} & K_{14} & . & . & K_{1n} \\ K_{21} & \boldsymbol{K_{22}} & K_{23} & K_{24} & . & . & K_{2n} \\ K_{31} & K_{32} & \boldsymbol{K_{33}} & K_{34} & . & . & K_{3n} \\ K_{41} & K_{42} & K_{43} & \boldsymbol{K_{44}} & . & . & K_{4n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ K_{n1} & K_{n2} & K_{n3} & K_{n4} & . & . & \boldsymbol{K_{nn}} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{K_{11}} & 0 & 0 & 0 & . & . & 0 \\ K_{21} & \boldsymbol{K_{22}} & 0 & 0 & . & . & 0 \\ K_{31} & K_{32} & \boldsymbol{K_{33}} & 0 & . & . & 0 \\ K_{41} & K_{42} & K_{43} & \boldsymbol{K_{44}} & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ K_{n1} & K_{n2} & K_{n3} & K_{n4} & . & . & \boldsymbol{K_{nn}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & K_{12} & K_{13} & K_{14} & . & . & K_{1n} \\ 0 & \mathbf{0} & K_{23} & K_{24} & . & . & K_{2n} \\ 0 & 0 & \mathbf{0} & K_{34} & . & . & K_{3n} \\ 0 & 0 & 0 & \mathbf{0} & . & . & K_{4n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & \mathbf{0} \end{bmatrix}$$

Eq 424

So, the equilibrium equation is written:

$$\{P\} = \{u\}\,[L]_{Lower} + \{u\}\,[L]^{Upper} \qquad \text{Eq 425}$$

$$\{u\}\,[L]_{Lower} = \{P\} - \{u\}\,[L]^{Upper}$$

Then, the iterative process is defined as:

$$\{u\}^{(k+1)}\,[L]_{Lower} = \{P\} - \{u\}^{(k)}\,[L]^{Upper} \qquad \text{Eq 426}$$

Where:

- $\{u\}^{(k)}$ is the kth iteration of {u}
- $\{u\}^{(k+1)}$ is the $(k+1)^{th}$ iteration of {u}
- The iteration process starts with an estimate.

The Gauss-Seidel method solves the left-hand side of Eq 426 for {u} using the previous value for {u} on the right-hand side. This is written as:

$$\{u\}^{(k+1)} = [L]_{Lower}^{-1}\left(\{P\} - [L]^{Upper}\{u\}^{(k)}\right) \qquad \text{Eq 427}$$

Taking advantage of the triangular form of [L], the elements of $\{u\}^{(k+1)}$ are computed sequentially using forward substitution:

For i = 1, 2, …, n

$$u_i^{(k+1)} = \frac{1}{K_{ii}}\left[f_i - \sum_{j=1}^{n} K_{ij}u_j^{(k+1)} - \sum_{j=i+1}^{n} K_{ij}u_j^{(k)}\right] \qquad \text{Eq 428}$$

The iteration process continues until the solution converges:

$$\frac{\left\|u^{(k+1)} - u^{(k)}\right\|}{\left\|u^{(k)}\right\|} \leq \varepsilon \qquad \text{Eq 429}$$

where ε is the tolerance for convergence control. The analyst must specify the maximum number of iterations to be performed, and a convergence criterion must be defined to determine when the solution converges. Since the iterative method must begin with an initial value, solvers typically use a preconditioning algorithm. However, there is no guarantee that a solution will result within the number of iterations specified by the user. For a large problem with a high number of degrees of freedom and, therefore, a large number of equations, the iterative method requires less memory than the direct method. Ultimately, the solution's accuracy depends on the convergence tolerance: a smaller tolerance will result in a more accurate solution but may take more iterations and, consequently, more computing time.

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